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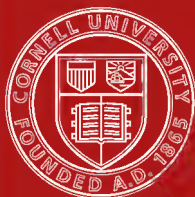
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STATICS AND DYNAMICS

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STATICS AND DYNAMICS

BY

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LONDON
LONGMANS, GREEN, AND CO.
AND NEW YORK : 15 EAST 16th STREET

1893

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PREFACE

THE OBJECT of this Elementary Text-book is to present both Statics and Kinetics in a combined form suitable for beginners. I have given the proof of the Parallelogram of Forces which is based on Newton's Laws of Motion, since that is the form which appears to me to be most intelligible to the student, and to follow most naturally from the Definition of Force; at the same time, Duchayla's proof is so interesting and ingenious, that it is entitled to a place in any book on dynamics; I have therefore added it in an Appendix.

Several chapters have been introduced dealing with subjects often omitted in elementary text-books—*e.g.* Couples, the General Conditions of Equilibrium, Virtual Work, and Normal Acceleration. I have endeavoured, when possible, to familiarise the student with more advanced methods of work, by introducing the idea of limits in the definitions of Variable Velocity and of Variable Acceleration which I use in the chapter on Normal Acceleration and the Hodograph.

A knowledge of the trigonometrical relations of one angle, such as is provided in my 'Introduction to Elementary Trigonometry,' is assumed; articles requiring

a more advanced knowledge of trigonometry, and chapters and articles which present difficulties to the beginner, are usually marked with an asterisk. Articles and paragraphs printed in smaller type are subsidiary to the more important matter printed in the ordinary type. Explanatory examples are worked out in each chapter.

The book is designed to meet the requirements of students preparing for the London University Matriculation, the Further Examinations for admission into Sandhurst and Woolwich, the first and second stages of the Examination in Theoretical Mechanics of the Science and Art Department, and the Oxford Local Examination for Senior Students. In the case of the following examinations, students may find it advisable to omit the parts noted :—

Cambridge Local Examination (Seniors) : Chapters XV. and XVI.

Cambridge Previous Examination (Additional Subjects) :
Articles 74 to 79, 91 to 93, 97 to 102 ; Chapters IX., X. ; Articles 181 to 186 ; Chapters XV. to the end.

Cambridge General Examination : Articles 74 to 79, 91 to 93, 97 to 102 ; Chapters IX. to the end.

My thanks are due (1) to Messrs. A. and C. Black, the publishers of the 'Encyclopædia Britannica,' for permission to make quotations from the articles on 'Mechanics,' 'Applied Mechanics,' and 'Gunnery,' and also to the authors of those articles ; but especially to Professor Tait, to whose teaching I am indebted for

much of what is best in my own ; (2) to the Publishing Committee of the Society for Promoting Christian Knowledge, for permission to quote from the late Professor Clerk Maxwell's 'Matter and Motion'—a book which every student of dynamics should read.

I am much indebted to Mr. H. C. Knott, of Selwyn College, for correcting the proof sheets, and for many valuable suggestions ; also to Mr. J. M. Dodds, of St. Peter's College, for much valuable criticism in the Introductory Chapter.

C. GELDARD.

CAMBRIDGE :

February 22, 1893.

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[Chapters and Articles marked with an asterisk
may be omitted by beginners.]*

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DYNAMICS



CHAPTER I

DEFINITIONS, UNITS, LAWS OF MOTION

1. Dynamics is the Branch of Natural Philosophy which treats of the action of Force upon Matter, and of the laws by which Motion is governed.

It may be divided into three parts—Statics, Kinematics, Kinetics.

Statics relates to forces which so act upon bodies as to keep them at rest.

Kinematics is the theory of motion considered without reference to the mutual action between bodies; and when this mutual action is taken into account it is called Kinetics.

2. In the above definitions the word **body** is used to denote any portion of **matter** of finite size. The conception of **matter** must be familiar to every intelligent person; it refers to that which is **concrete** as opposed to that which is **abstract**.

A portion of matter indefinitely small in all its dimensions is called a **Particle**.

A particle is then a material point of such small linear dimensions that it presents no difference in aspect however it be turned about itself; and is such, that when it is acted on by a force, the effect will be that of translation only; that is to say, it will begin to move in a straight line, and its rotation about itself may be neglected.

Bodies of finite size may be supposed to consist of an infinite number of particles.

3. **Force is any action between two bodies which changes or tends to change the physical relation between them as regards rest or uniform motion in a straight line.**

4. Our first impression of force is the result of a muscular effort in the act of pushing or pulling a body ; but when any other agent produces the same or a similar effect on a body, it is said to exert force. This effect is always reciprocal—equal and opposite ; the force which A exerts on B is equal and opposite to that which B exerts on A ; *e.g.* a book is resting on a table ; the pressure of the table on the book is equal and opposite to the pressure of the book on the table. Again, a horse pulls a cart ; the pull of the horse on the cart is equal and opposite to the pull of the cart on the horse ; the former makes the cart move, the latter is counterbalanced by the muscular exertion of the horse. Thus force is always the action between two bodies ; it is, however, often convenient to consider only its effect upon one of them, without considering the agency by which it is produced ; it may then be considered as **any cause which changes or tends to change a body's existing state of rest or uniform motion in a straight line.**

5. The effect of a force in changing a body's shape is called a *strain*. The consideration of this effect of force is of too difficult a nature to be attempted in an elementary work.

The effect of a force in changing a body's motion may be of two kinds—a **translation** and a **rotation**. When a force acts upon a particle, the effect is that of translation only (see definition of Particle in Art. 2) ; but when it acts upon an extended body, in addition to the motion of translation, there is also usually a rotation about an axis within the body.

6. There are various ways in which Force acts, for which we have different names. Push, or **Pressure**, is the name given to the force exerted at the point or points of contact when one body presses against another. Pull, or **Tension**, is the name given to the force exerted by one body on another with which it is connected by means of a string or rod. The nature of a Force, however, is always the same, whatever be the name which is given to it.

7. There is a third kind of force—which is apparently

neither pressure nor tension—which is exerted by one body on another in no way connected with it, and which we call **Attraction**. The most familiar example of this force is the attraction which the earth exerts upon all matter, called the Force of **Gravity**. The measure of the force of gravity on any body is called the body's **Weight**.

8. There are three elements of a force which must be known before the effect of the force on a body can be determined ; they are as follows :—

(1) The *direction* of the force ; that is, the direction in which a particle would begin to move under the action of the given force only.

(2) A point in the line of action of the force, usually called the *point of application* of the force.

(3) The *magnitude* of the force.

9. It is immaterial what point in the line of action of the force is given ; the effect of the force on the body will be the same if we suppose it to act at one point in its line of action as at another, provided that they are rigidly connected with the body. This is called the **Principle of the Transmissibility of Force**.

The following illustration will assist in explaining it : Suppose a weight attached to the hand by means of a thread, whose weight may be neglected. The muscular effort required to support the weight is always the same, at whatever point of the thread we suppose the weight to be attached.

It will be found that the motion of translation of a body will not be changed by changing the line of action of the force, provided the direction of the force remain the same ; but when we consider the rotation of the body, it is necessary that we should know the line of action as well as the direction of the force.

Geometrical Representation of Forces.

10. Forces may be completely represented by straight lines. For a straight line is completely ascertained when we know :

(1) The point from which it is drawn.

(2) The direction in which it is drawn.

(3) Its length.

Thus a straight line may be drawn

(1) From any point which may represent a point in the line of action of the force.

(2) In the direction of the force.

(3) Of such a length as to contain as many units of length as the force contains units of force.

It will then represent the force completely.

11. In Art. 1 we used the terms 'rest' and 'motion.' These terms are relative: the state of absolute rest is unknown. A body is said to be at rest when it is relatively at rest; that is, when it continually occupies the same position with reference to other objects; and it is said to be in motion when its position is changing; that is, when it occupies different positions at different times among the objects to which it is referred.

Consider, for example, the state of a stone apparently at rest on the surface of the earth. Its actual motion in space is very complicated; for the earth is rotating on its own axis once a day; it is moving in its orbit round the sun once a year; and, lastly, the whole solar system is moving towards a certain point of the heavens.

Hence the actual motion in space of any point on the earth's surface is a combination of at least three motions, although it is at rest with reference to the earth.

Again, consider the case of a stone resting on a table in a railway carriage which is moving.

The stone is at rest with reference to the table and the carriage, but the stone and the table and the carriage are all moving in the same manner with reference to the earth. It is possible, again, for the stone to be moving on the table and the table in the carriage in such a manner that the position of the stone in the carriage remains the same; that is, it will still be at rest with reference to the carriage.

12. Before proceeding further with the definitions of the physical quantities which we employ in the Science of Dynamics, it will be well to consider briefly the method of

measuring quantities and the choice of a unit of measurement.

In order to measure any quantity we choose some unit of the *same kind* as the quantity under consideration, and then compare the given quantity with the unit chosen ; the *measure* of the quantity is the number of times it contains the unit chosen ; the greater the unit is, the smaller will be the measure. Thus, if m, m' be the measures of the same quantity in terms of the units U, U' respectively,

$$m U = m' U'$$

hence the measure of a quantity varies inversely as the unit chosen.

$$\begin{aligned} \text{e.g. } 60 \text{ pence} &= 5 (12 \text{ pence}) = 5 \text{ shillings} \\ &= \frac{5}{20} (20 \text{ shillings}) = \frac{1}{4} \text{ of } \pounds 1 \end{aligned}$$

$$\text{and } 6 \text{ feet} = \frac{6}{3} (3 \text{ feet}) = 2 \text{ yards}$$

$$\text{or} = 6 (12 \text{ inches}) = 72 \text{ inches}$$

$$\text{thus } \frac{6}{72} = \frac{1}{12} = \frac{1 \text{ inch}}{1 \text{ foot}}.$$

The measurement of all dynamical quantities depends upon the three primary or fundamental conceptions of *space, time,* and *matter*, for the measurement of which we have three fundamental units which are perfectly arbitrary, and which vary in different countries.

The unit of length—or linear space—employed in England is one foot, which is equal to one-third of the length of a piece of platinum which, as well as various duplicates, is carefully preserved in London, and is called the Standard Yard.

The unit of area is the area of a square surface whose sides are each one foot long, and is called a 'square foot.'

The unit of volume is the space contained in a cubical box whose edges are each one foot long, and is called a 'cubic foot.'

In France the unit of length is one centimetre, which is the one-hundredth part of a metre—a metre being an arbitrary

length of platinum which is kept in Paris, chosen originally as the length of the seconds pendulum.

$$\begin{aligned}\text{One metre} &= 39\cdot37 \text{ inches approximately,} \\ &= 1\cdot09 \text{ yard,}\end{aligned}$$

$$\text{One centimetre} = \cdot3937 \text{ inches.}$$

The unit of time is the same in both countries : it is one second.

$$\begin{aligned}\text{One second} &= \frac{1}{60} (\text{a minute}) = \frac{1}{60 \times 60} (\text{an hour}) \\ &= \frac{1}{60 \times 60 \times 24} (\text{a day}),\end{aligned}$$

and a day is the length of time it takes the earth to make a complete revolution on its axis. Hence a second is the time during which the earth revolves through an angle of $15''$.

The English unit of matter is the quantity of matter contained in a lump of platinum which is kept in London, and which is called a 'pound.'

The French unit of matter is a different quantity, and is called a 'gramme.'

All dynamical quantities will be measured in terms of these three fundamental units : in England, in terms of a foot, a second, and a pound ; in France, in terms of a centimetre, a second, and a gramme.

The latter system of measurement is called the C.G.S. system, and is largely employed in scientific investigations in England as well as in France.

13. When a body is in motion it is said to have a certain *velocity*, which measures the rate at which it is moving in a straight line.

If the motion is not in a straight line the rate of motion is sometimes called *speed*.

' This velocity may be *constant* or *variable*.

A body is said to move with constant velocity when it passes over equal distances in equal intervals of time ; the velocity is then measured by the number of units of length which the body passes over in one unit of time.

Thus, if a body passes over s units of length in t units of time, it will pass over $\frac{s}{t}$ units of length in one unit of time ; if we call this velocity v ,

$$v = \frac{s}{t}.$$

The unit in which v is measured is called the 'unit of velocity,' and is the velocity of a body which passes over one unit of length in one unit of time ;

$$\text{thus } v = \frac{s}{t} \text{ (units of length) per unit of time.}$$

The measure $\frac{s}{t}$ is always the same, however great s and t may be, provided that the velocity remains constant.

The unit of velocity in English units is the velocity of a body which passes over one foot in one second ; thus, if a body pass over s feet in t seconds,

$$\text{its velocity} = \frac{s}{t} \text{ feet per second.}$$

The unit of velocity in French units is the velocity of one centimetre per second.

14. A body is said to move with variable velocity when it passes over unequal distances in equal intervals of time.

The velocity of the body may be different at every point of its course, but its *average velocity* during any time will be equal to the velocity of a body which moves over the same distance in the same time with constant velocity.

15. We have noticed that when a body is moving with constant velocity the measure of that velocity ($v = \frac{s}{t}$) is the same, however large t may be ; and this formula is always true of the average velocity of a body during any time. Now, when a body is moving with variable velocity, the smaller we take t the more nearly does the average velocity during that time approximate to the actual velocity at any instant during

that time ; and when t is indefinitely small, the actual velocity and the average velocity are equal.

Hence the actual velocity of a body, whether it is moving uniformly or not, may be measured at any instant by the indefinitely small space described in a small portion of time including the instant, divided by the indefinitely small time required to describe it.

16. When a body is moving with variable velocity in a straight line, **the rate at which the velocity changes is called Acceleration.** Acceleration may, like velocity, be *constant* or *variable*.

A body is said to be moving with constant acceleration when it receives equal increments of velocity in equal intervals of time ; its acceleration is said to be variable when it receives unequal increments of velocity in equal intervals of time.

When acceleration is opposite to the direction of motion it is sometimes called a 'retardation.'

17. When a body is moving with constant acceleration, this acceleration is measured by the number of units of velocity gained in a unit of time.

Hence, if a body gain v units of velocity in t units of time, it will gain $\frac{v}{t}$ units of velocity in one unit of time ; if we call this acceleration a ,

$$a = \frac{v}{t} \text{ units of velocity per unit of time.}$$

The unit of acceleration is therefore an increment in velocity of one unit of velocity in one unit of time.

In English units the unit of acceleration is an increment in velocity of one unit of velocity per second ; that is, of one foot per second, per second.

Thus the above equation would be read :

$$a = \frac{v}{t} \text{ feet per second, per second.}$$

Similarly the unit of acceleration in French units is an acceleration of one centimetre per second, per second.

When the acceleration of a body is variable, it is measured at any instant by the indefinitely small change in velocity in a small portion of time including the instant, divided by the indefinitely small time during which the change takes place.

18. Hitherto we have confined our attention to explaining the meaning of certain terms used in the Definitions of the Subject. We shall now consider the relation between force and matter more fully, dealing only with those very small portions of matter which we have called 'particles.'

It must be borne in mind that these particles contain different quantities of matter, but their size and shape need not be considered.

19. We are already prepared by our definition of Force for Newton's *First Law of Motion*. This may be stated as follows :—

'Every body remains in its state of rest or of uniform motion in a straight line, unless it is compelled by external forces to change that state.'

This property of Matter, in virtue of which it requires force to change its state of rest or uniform motion, is called its '**Inertia**,' and the first law of motion is sometimes called the **Law of Inertia**.

20. This property of Inertia is familiar to all of us—*e.g.* : We are sitting in a railway carriage at rest, with our faces towards the engine ; the carriage receives an impulse from the engine forwards, we feel impelled against the back of the seat behind us. We were at rest and tended to remain at rest after the carriage has begun to move.

Again, let us watch a pendulum, consisting of a bullet tied to a thread and suspended from the roof of a railway carriage going at full speed, the pendulum being at rest with reference to the carriage ; directly the brakes are put on, and the train commences to slacken speed, the pendulum begins to oscillate in the direction of the train's motion, and then back.

Again, suppose we roll a ball along a road : the ball is quickly brought to rest on account of its friction with the ground and the resistance of the air. If we now take a very smooth round ivory ball and roll it on very smooth ice, it will

continue to move for a very much longer time than on the road ; and we argue that if the friction between the ball and the surface, and the resistance of the air, could be got rid of altogether, the ball would go on moving for ever with uniform velocity in a straight line.

21. Let us now consider another example. Suppose we jump out of a carriage which is in motion : we remain in this state of motion until our feet touch the ground. We then feel a sudden impulse in the direction of motion, because our feet are brought to rest by contact with the ground, while our body still retains its original state of motion.

A *heavy* man will feel this impulse much more severely than a *light* man ; the latter will often be able to keep on his feet when the former falls prostrate on the ground.

We see in this case that the inertia of a body depends upon and is a measure of the quantity of matter it contains, or, as it is usually called, its 'mass.'

As a further illustration of this, consider the following case :—A croquet ball and a cannon ball of the same size are at rest on the ground, and we give a kick of equal strength to each of them : the croquet ball will begin to move with a much greater velocity than the cannon ball, although the impulse given to each is the same. Again, in catching or stopping a cricket ball, we exert a much greater force than in stopping a woollen ball of the same size which reaches us with the same velocity.

22. From these examples we see that **the quantity of motion** in a moving body depends upon two things : (1) the **velocity**, (2) the **mass**.

23. The mass of a body depends upon two things—its volume or cubical content, and the material of which it is composed. The quality of the material is measured by a number which we call its 'density.'

If m be the mass of a body, v its volume, and ρ its density,

$$m = v \cdot \rho.$$

If $v = 1$, $m = \rho$; thus the density of a substance is the mass of the unit volume of the substance.

24. As already stated in Art. 12, the **British Standard unit**

of mass is one pound avoirdupois, and is a certain mass of platinum kept in the Standards Office of the Board of Trade.

N.B.—**The mass one pound** must be carefully distinguished from **the weight of one pound**; the former is merely a lump of matter, the latter measures the force with which such a lump of matter is attracted by the earth.

The French unit of mass is one gramme; this is the one-thousandth part of a mass of platinum kept in the French archives. The gramme was originally chosen as being the mass of a cubic centimetre of pure water at 4°C.

One pound is equivalent to 453·593 grammes.

25. Bodies are said to be of **equal mass** when equal impulses generate in them equal velocities; or when equal forces acting upon them for the same time generate in them equal velocities.

For example: place three balls made of wood, stone, and lead on a smooth table and let the mass of each be 1 lb.; now apply simultaneously to each an equal impulse by means of a spring; it will be found that they reach the other side of the table at the same time; that is, they move with equal velocities.

Again, place three balls on the table as before, and let their masses be 1 lb., 2 lbs., 3 lbs.; now apply an equal impulse to each; it will be found that the times they take to get to the other side of the table will be in the ratio of 1 : 2 : 3, which means that their velocities are in the ratio of 3 : 2 : 1. This will not be exactly true, on account of the friction of the table.

Hence we say that **quantity of motion is measured by the product mass $\times$ velocity**.

This is usually called **Momentum**.

Hence, if **M** stand for the momentum of a body of mass **m** moving with a velocity **v**,

$$M = m \cdot v.$$

The unit of momentum is the momentum of unit mass moving with the unit velocity. In English units it is the momentum of a mass of one pound moving with a velocity of one foot per second.

26. We find, then, that when the same force acts for the

same length of time upon particles of different mass, different velocities are generated, but in every case the momentum generated is the same.

Hence, **force may be measured by the momentum generated in a unit of time.**

Thus, if v, v' be the velocities in the same straight line of a particle of mass m at the beginning and end of a time t ,

$$\begin{aligned} mv &= \text{momentum at the beginning of the time} \\ mv' &= \text{,, ,, end ,, ,,} \\ mv' - mv &= \text{change of momentum in } t \text{ seconds} \\ \frac{mv' - mv}{t} &= \text{,, ,, in 1 second.} \end{aligned}$$

If we call the moving force in this direction F ,

$$\begin{aligned} F &= \frac{mv' - mv}{t} \\ &= \frac{m(v' - v)}{t} \\ &= m \cdot a \end{aligned} \quad (\text{Art. 17}),$$

where a is the acceleration produced by the force.

27. The equation $F = ma$, states in an algebraical form Newton's *Second Law of Motion*, which is as follows :—

Change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts; provided that the unit of force is taken to be that force which produces the unit of acceleration in the unit of mass.

28. **The British Absolute unit of Force is that force which, when acting on a mass of one pound, communicates to it an acceleration of one foot per second, per second.**

This unit of Force is called a **Poundal**.

29. We shall see later that when a body is allowed to fall freely *in vacuo* it moves with a constant acceleration which we will denote by g . This acceleration is due to the force of gravity; that is, it is the acceleration caused by the constant force of the body's weight acting on its mass.

The value of g varies at different points on the earth's surface. Its value in England in English units is 32.2 approximately.

Hence the weight of one pound acting on a mass of one pound produces an acceleration of 32.2 feet per second, per second.

But the unit force is that which acting on one pound produces an acceleration of one foot per second, per second,

$$\text{therefore the unit force} = \frac{1}{32.2} \times (\text{weight of one pound})$$

$$= \text{the weight of half an ounce approximately.}$$

Or, calling the acceleration due to gravity g , we may say that

$$g \cdot \text{Poundals} = \text{weight of one pound.}$$

In Statics it is customary to consider the **weight of one pound** as the unit of Force. This unit of force varies at different points on the earth's surface, whereas the Poundal is always the same at every point. Hence the Poundal is called the **Absolute unit of Force**.

30. In the French system of measurement, where the units of length, mass, and time are one centimetre, one gramme, and one second respectively, the unit of Force is called a **Dyne**.

Hence a Dyne is the force which, acting on a mass of one gramme, produces an acceleration of one centimetre per second, per second.

Now 32.2 feet = 980 centimetres approximately, so that 980 is the acceleration due to the action of the weight of one gramme on the mass of one gramme.

$$\text{Hence } 980 \text{ Dynes} = \text{weight of one gramme.}$$

If we compare the units of Force in the two systems we shall find that

$$\text{one Poundal} = 13825 \text{ Dynes.}$$

For $1 \text{ lb.} = 453.593 \text{ grammes}$, and $1 \text{ foot} = \frac{12}{0.3937} \text{ cms.}$ approximately.

$$\therefore \text{one poundal} = \frac{453.593 \times 12}{0.3937} \text{ dynes.}$$

31.* The equation $F = m \cdot a$ is true of all finite forces acting for an appreciable time.

It is also true for blows or impulses which produce a finite change of motion in an indefinitely short time.

Thus, if F be the force acting on mass m , a the acceleration generated, τ the small time during which F acts, and v and v' the velocities before and after the blow, we have

$$\begin{aligned} v' - v &= a \cdot \tau \text{ and } F = m \cdot a \\ \therefore F \tau &= m \cdot a \cdot \tau \\ &= m (v' - v). \end{aligned}$$

Thus, the product $F \cdot \tau$ measures the change in momentum caused by the blow, that is, the impulse, just as $F \cdot t$ measures the effect of the force F during a finite time t .

It must, of course, be borne in mind that τ is often so small as to be inappreciable, but F may be so large that the product $F \cdot \tau$ is finite and is measured by the change of momentum which is produced by the blow.

32. Newton's *Third Law of Motion* is as follows :—

'To every action there is always an equal and opposite reaction.'

The truth of this law was illustrated when we considered the meaning of Force (Art. 4). We then saw that there could be no force without a reaction.

Take, for example, the recoil of a gun when it is fired off.

$$\left. \begin{array}{l} \text{Momentum generated} \\ \text{in the bullet} \end{array} \right\} = \left\{ \begin{array}{l} \text{Momentum of the} \\ \text{gun's recoil.} \end{array} \right.$$

$$\begin{aligned} \text{Mass of projectile} \times \text{vel. of projection} &= \text{mass of gun} \\ &\times \text{vel. of recoil.} \end{aligned}$$

This law embodies the Principle of the Conservation of Energy.

33. For the sake of impressing the three Laws of Motion on the memory of the student we will again state them.

Law I. Every body remains in its state of rest or of uniform motion in a straight line, unless it is compelled by external forces to change that state.

Law II. Change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.

Law III. To every action there is always an equal and opposite reaction.

34. Of these three laws, the second is at present by far the most important.

It involves the following principles :—

If there is no force, there is no change of momentum.

The effect of a force is exactly the same whether the body be at rest or in motion—it is always measured by the momentum generated, and this change of momentum is in the direction of the force.

If two or more forces act upon a body at the same time, the change of motion due to each is independent of the others, and may be considered without reference to them.

Ex. 1.—Find the force which acting on a mass of 4 lbs. generates a velocity of 20 feet per second in 10 seconds.

$$\text{Here the acceleration} = \frac{20}{10} = 2$$

$$\therefore F = m a = 4 \times 2 = 8 \text{ poundals.}$$

Ex. 2.—Find the velocity generated by a force of 7 poundals in a mass of 1 stone after it has acted for 1 minute.

$$F = m a$$

$$7 = 14 \cdot a$$

$$\therefore a = \frac{1}{2}$$

Velocity generated in one minute

$$= a \cdot t = \frac{1}{2} \times 60$$

$$= 30 \text{ feet per second.}$$

Examples.

1. Express a velocity of 60 miles an hour in feet per second.
2. A man walks 30 miles in 8 hours. What is his velocity in feet per second?
3. Assuming an inch is 2.5 centimetres, express a velocity of 4 miles an hour in terms of centimetres and seconds.
4. Express in centimetres per second a velocity of 4 kilometres per hour.

5. A man travels a feet in b seconds. Find the distance he will travel in a minute, and the time he will take to travel 100 yards.

6. What is the velocity of a point on the earth's equator, the radius of the earth being 4,000 miles?

7. A man 6 feet high walks at the rate of 4 miles an hour away from a lamp 10 feet above the ground. Find the velocity of the end of his shadow.

8. A body starts from rest and is subject to a uniform acceleration. It acquires a velocity of 20 feet per second at the end of the fourth second. What is the acceleration?

9. A body moves over 1, 3, 5, 7 feet during the 1st, 2nd, 3rd, and 4th seconds respectively. Find the average velocity.

10. A body moving with uniform acceleration has a velocity 10. A minute later its velocity is 40. What is the acceleration? Have the units employed any effect on the answer?

11. Compare the velocities 66 feet per second and 40 miles per hour.

12. What is the momentum of a body whose mass is 5 lbs. moving with a velocity of 2 feet per second? What force is required to produce this momentum in one second? What force is required to produce this momentum in the hundredth part of a second? How many pounds weight is this latter force?

13. Compare the momentum of a mass of 100 lbs., moving with a velocity of 1,000 feet a second, with that of a mass of 1 lb. moving with a velocity of 30 feet a minute.

14. What force is required to reverse the motion of a mass of 1 lb. moving with a speed of 100 feet per second, supposing the whole change to take place in one-tenth of a second?

15. A force of 20 poundals acts on a mass of 4 lbs. What is the acceleration? Determine the velocity at the end of one minute if it starts from rest.

16. A force of 1 poundal acts on a mass of 1 ounce. Determine the velocity acquired in 5 seconds.

17. A force acts on a mass of 1 cwt. and communicates a velocity of 20 feet per second in 4 seconds. Find the force.

18. A force of 56,000 poundals acts on a truck on smooth rails and communicates to it a velocity of 300 feet per second in 1 minute. Find the mass of the truck in tons.

CHAPTER II

PARALLELOGRAM OF VELOCITIES AND FORCES

35. We shall now consider how to represent the velocity of a point and the composition of two or more velocities.

The velocity of a point is known when its direction and magnitude are known. We can then completely represent the velocity of a point O by a straight line OA .

For let v be the velocity of a point O moving in a direction making an angle θ with a fixed straight line OX . Now if OA be drawn, making the same angle θ with OX , and containing as many units of length as the point has units of velocity, OA completely represents the velocity of the point O .

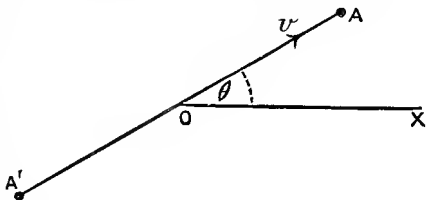


FIG. 1.

The unit of length in which OA is measured may be any unit that is convenient, provided always we keep our unit unchanged during the consideration of any mechanical theorem. If the velocity of a point in one direction be called v , an equal velocity in the opposite direction will be $-v$.

Thus, if OA represent the velocity v , OA' equal and opposite to OA will represent a velocity $-v$.

36. A body may have at the same instant velocities in two or more different directions quite independent of each other.

For example, if a person walks from one side of a railway carriage in motion to the other, his apparent motion in the carriage is the same as if it were at rest, but his actual motion referred to the surface of the earth will be compounded of the two motions—that of the carriage and his own motion in

the carriage ; and if these two motions are uniform, his actual motion must also be uniform.

Def.—The velocity which is equivalent to two or more velocities is called their **Resultant**, and with reference to this resultant each of the single velocities is called a **Component**.

We shall now show how the resultant of two velocities may be found.

Parallelogram of Velocities.

37. If the two simultaneous velocities of a moving point be represented in magnitude and direction by two straight lines drawn from a point, and if the parallelogram having these two lines for adjacent sides be completed, then the diagonal of the parallelogram which passes through the point will represent the resultant velocity in magnitude and direction.

Let u and v be the two simultaneous velocities of the point, and let them be represented in magnitude and direction by the straight lines OA , OB .

Complete the parallelogram $OACB$, OC shall represent the resultant velocity.

We may imagine the point to move from O to A in a unit of time, while the page of the book moves parallel to OB with a velocity represented by OB . Thus, while the point moves from O to A the line OA will move into the position BC ; and owing to these two simultaneous velocities the moving point will arrive at C in the unit of time.

Now, since the two simultaneous velocities are constant in magnitude and direction, the actual velocity of the point from O to C must be constant in magnitude and direction ; that is, its path from O to C must be in a straight line. Hence OC represents in direction and magnitude the path described by O in the unit of time.

$\therefore OC$ represents the resultant of the two simultaneous velocities represented by OA , OB .

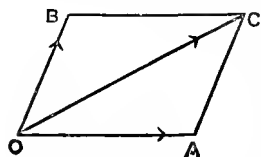


FIG. 2.

This proposition holds of any two simultaneous velocities which a point may have, whether it has other independent velocities or not. It is also true of two variable velocities by the extended definition of velocity in Art. 15.

38. The acceleration of a point is the rate at which its velocity is changing, and is measured by the number of units of velocity the point gains in a unit of time (Art. 17).

Like velocity, it may be represented completely in magnitude and direction by a straight line, and may be positive or negative according to the direction in which the straight line is drawn.

Thus, if a positive acceleration, that is, an acceleration in the direction in which the point or particle is moving, be represented by a straight line in one direction, an acceleration equal and opposite to this, that is, a retardation, will be represented by an equal straight line, in the opposite direction.

Acceleration may likewise be compounded and resolved in the same way as velocities.

We shall thus have the **Parallelogram of Accelerations**, which may be stated as follows :—

39. If a point have simultaneously two accelerations which are represented in magnitude and direction by two straight lines drawn from the point, and if the parallelogram having these two lines for adjacent sides be completed, the diagonal of the parallelogram drawn through the point will represent the resultant of the two accelerations in magnitude and direction.

For let OA , OB represent the two accelerations of the point.

Complete the parallelogram $OACB$, and join OC .

Then OA , OB will represent in magnitude and direction the velocities due to each acceleration which are added in a unit of time.

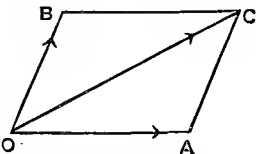


FIG 3.

Therefore, by the Parallelogram of Velocities, OC represents in magnitude and direction the resultant velocity added in that unit of time.

Hence OC represents in magnitude and direction the resultant acceleration of the two simultaneous accelerations represented by OA , OB .

40. We have seen (Art. 26) that Forces are measured by the momenta which they produce in a unit of time ; that is, they are measured by the product, **mass $\times$ acceleration**. Hence, if two or more forces act upon particles of equal mass, they will be proportional to the accelerations which they produce in them, since the mass moved is the same for each force.

The Second Law of Motion further states (Art. 34) that, if two or more forces act upon a body at the same time, the change of momentum due to each is independent of the others, and is the same as if that force acted alone.

We may therefore say that, if several forces act simultaneously on the same particle, each force will be proportional to the acceleration which it produces in it.

Definition.—The **resultant** of two or more forces is the single force whose effect is equivalent to the combined effect of the forces, and with reference to this resultant each of the original forces is called a **Component**.

Forces also may be represented completely by straight lines (Art. 10).

We will now state and prove the **Parallelogram of Forces**.

41. If two forces acting on a point be represented in magnitude and direction by two straight lines drawn from the point, and if the parallelogram having these two lines for adjacent sides be completed, the diagonal of the parallelogram drawn through the point will represent the resultant of the two forces in magnitude and direction.

Let OA , OB represent in magnitude and direction the two forces which act on a particle at O .

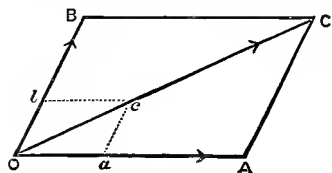


FIG. 4

Complete the parallelogram $OACB$, and join OC .

Then since OA , OB represent the forces acting on O , they will also be proportional to the accelerations which those forces produce in O (Art. 40).

Hence if Oa , Ob represent the accelerations produced by the forces represented by OA , OB ,

$$OA : Oa :: OB : Ob,$$

and if the parallelogram $Oacb$ be completed, the point c must lie on the diagonal OC (Euclid VI. 4). And by the parallelogram of accelerations, Oc represents the resultant of the two accelerations represented by Oa , Ob .

Now, since the mass acted upon remains the same, the resultant force must have the same ratio to the resultant acceleration as either of the component forces bears to the acceleration which it produces.

$$\text{But } OC : Oc :: OA : Oa :: OB : Ob,$$

$\therefore OC$ must represent the resultant force in magnitude and direction.

42. The results of the last article and of those which precede it are of the very greatest importance. Before proceeding further, the student should again read over the previous articles of this chapter, and satisfy himself that he thoroughly understands the argument contained in them; noticing how we derive the Parallelogram of Accelerations from the Parallelogram of Velocities, and thus, with the help of the Second Law of Motion, are enabled to deduce the Parallelogram of Forces.

CHAPTER III

STATICS

Forces at a Point

43. THE Science of Statics consists of the investigation of all cases in which the external forces which act upon a body are so related that the body remains at rest, or moves uniformly in a straight line.

When a body is apparently at rest, or moves with a constant velocity in a straight line, the external forces, if any, which act upon it balance each other, or are in equilibrium. Thus, when a steam engine moves with uniform velocity in a straight line, the various forces which act upon it—*e.g.* the force exerted by the engine, the friction of the rails, the resistance of the air, the weight of the engine, and the pressure of the rails—are in equilibrium.

As stated in Art. 11, a body at rest with respect to the surface of the earth has in space a motion which is neither uniform nor in a straight line. In an elementary work this motion cannot be taken into account, nor does it affect our results to any appreciable extent. Hence in statical investigations we shall not take the motion of the earth into account; that is to say, that when a body is at rest with respect to the surface of the earth we shall regard it as absolutely at rest.

Thus, if the steam engine mentioned above be at rest on smooth level rails, we say that the weight of the engine and the pressure of the rails are in equilibrium; that is, they are equal and in opposite directions.

44. It is obvious that when a system of forces so acts on a body or material system as to be in equilibrium, it is equivalent to no force at all; consequently the result will be the same if the forces act in the same manner on any other body or material system, whatever its mass may be.

This is the reason why the consideration of mass does not enter into statical investigations.¹

For the same reason also, any system of forces in equilibrium may be introduced into any other system of forces acting on a body, or removed from such a system, without disturbing its effect.

We shall use the above axiom most frequently in introducing or removing the equal and opposite forces.

45. In Statics, forces are usually measured by the weight which they will support, and the weight of 1 lb. is taken as the unit force (Art. 29). This is usually written 1 lb. wt. Similarly a force equal to the weight of P lbs. is called P lb. wt. Some writers on the subject talk about forces of P lbs. and Q lbs.; this is misleading, since a pound avoirdupois is merely a lump of matter which we take as our unit of mass (Art. 23). Generally, we shall talk of forces P and Q , 'or' 3 and 5, &c., without designating the unit in which these forces are measured.

46. *To find the magnitude and direction of the resultant of two forces P and Q , acting on a particle and inclined to each other at an angle θ .*

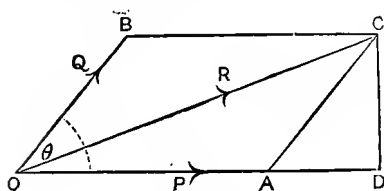


FIG. 5.

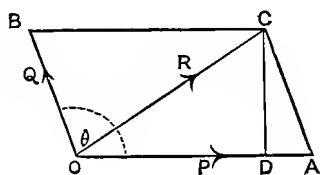


FIG. 6.

Let OA , OB represent the two forces P and Q in magnitude and direction so that the angle $AOB = \theta$.

Complete the parallelogram $OACB$ and draw CD perpendicular to OA or OA produced.

Then by Euclid II. 12 and 13,

$$OC^2 = OA^2 + AC^2 \pm 2 OA \cdot AD$$

according as the angle CAO is obtuse or acute.

¹ Clerk Maxwell, *Matter and Motion*, p. 46.

Now if $\angle C A O$ be obtuse, $C A D = \theta$;

$$\therefore A D = A C \cos \theta;$$

and if $\angle C A O$ be acute, $C A D = 180^\circ - \theta$;

$$\therefore A D = A C \cos C A D = - A C \cos \theta.$$

$\therefore$ whether $C A O$ be obtuse or acute,

$$O C^2 = O A^2 + A C^2 + 2 O A \cdot A C \cos \theta.$$

If we call the resultant force R ,

$$R^2 = P^2 + Q^2 + 2 P \cdot Q \cos \theta,$$

which determines the magnitude of the resultant. This result can also be obtained at once by substituting the proper values in the result of Article 29 of the author's 'Introduction to Trigonometry.'

To determine the direction of the resultant.

Let the angle $C O A = \phi$

$$\tan \phi = \tan C O D$$

$$= \frac{C D}{O D}$$

$$= \frac{C D}{O A \pm A D} \left\{ \begin{array}{l} \text{according as the angle } O A C \\ \text{is obtuse or acute.} \end{array} \right.$$

$$= \frac{A C \sin \theta}{O A + A C \cos \theta} \left\{ \begin{array}{l} \text{whether the angle is} \\ \text{obtuse or acute.} \end{array} \right.$$

$$= \frac{Q \sin \theta}{P + Q \cos \theta}.$$

Corollary.—If $\theta = 90^\circ$ $C A O = 90^\circ$

$$\therefore R^2 = P^2 + Q^2 \quad (\text{Euclid I. 47.})$$

This can also be obtained from the above equation by putting $\cos \theta = 0$.

Corollary.—If one of the forces Q remains constant, and the other P increases, the angle between R and P diminishes. This is obvious from the above formula for $\tan \phi$, or it can be easily proved geometrically

Example.

Find the magnitude of the resultant of two forces, 9 poundals and 7 poundals, at an angle of 120° .

$$\begin{aligned} R^2 &= P^2 + Q^2 + 2 P \cdot Q \cos \theta \\ &= 9^2 + 7^2 + 2 \cdot 9 \cdot 7 \left(-\frac{1}{2}\right) \\ &= 130 - 63 \\ &= 67 \end{aligned}$$

$\therefore$ Resultant force = $\sqrt{67}$ poundals.

To find the magnitude and direction of the resultant of two forces 5 and 4 acting at an angle of 45° .

$$\begin{aligned} \text{Resultant force} &= \sqrt{5^2 + 4^2 + 2 \cdot 5 \cdot 4 \cdot \frac{1}{\sqrt{2}}} \\ &= \sqrt{25 + 16 + 20\sqrt{2}} \\ &= \sqrt{41 + 20\sqrt{2}} \\ &= 8.3 \text{ approximately.} \end{aligned}$$

If ϕ be the angle it makes with the force 5,

$$\tan \phi = \frac{4 \sin 45^\circ}{5 + 4 \cos 45^\circ} = \frac{2\sqrt{2}}{5 + 2\sqrt{2}} = .36 \text{ approx.}$$

$$\therefore \phi = \tan^{-1} .36$$

$$\text{Hence } \phi = 19^\circ 48' \text{ approx.}$$

In working examples it is unnecessary to calculate the surds.

The student may leave the results of the last example in the form

$$\left\{ \begin{array}{l} R = \sqrt{41 + 20\sqrt{2}} \\ \tan \phi = \frac{10\sqrt{2} - 8}{17} \end{array} \right.$$

47. It is obvious that, if the two forces are acting in the same direction and in the same straight line, the resultant force is equal to their sum; and if they are in opposite directions in a straight line, the resultant is their difference, that is, their *algebraical* sum.

This also appears from the formula

$$R = \sqrt{P^2 + Q^2 + 2 P Q \cos \theta}$$

putting $\theta = 0^\circ$ $R = \sqrt{P^2 + Q^2 + 2 P Q}$
 $= P + Q$

putting $\theta = 180^\circ$ $R = \sqrt{P^2 + Q^2 - 2 P Q}$
 $= P - Q.$

If two equal forces act on a particle in opposite directions in a straight line, the resultant is zero, and the motion of the particle is unchanged.

48. *The resultant of two forces acting at a point acts nearer to the greater force.*

With the figure of Art. 46, suppose P greater than Q.

Then O A is greater than O B
i.e. „ A C.

But the greater side is opposite the greater angle,

$\therefore$ the angle A O C is less than the angle A C O
i.e. „ „ C O B;

$\therefore$ the resultant acts nearer to P than it does to Q.

The resultant of two forces diminishes as the angle between the forces increases.

This is at once apparent from the formula

$$R^2 = P^2 + Q^2 + 2 P Q \cos \theta,$$

for as θ increases from 0° to 180° , $\cos \theta$ diminishes from $+1$ to -1 .

$\therefore$ R diminishes from $P + Q$ to $P - Q$,

or it may be proved geometrically with the help of the figures of Article 46.

As B O A increases, O A C decreases,
 and since O A, A C, remain the same,

the length of OC decreases as the angle OAC decreases (Euclid I. 24),

$\therefore$ the resultant decreases as the angle AOB increases.

From the Parallelogram of Forces we may at once deduce the Triangle of Forces.

Triangle of Forces.

49. If three forces acting on a particle can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.

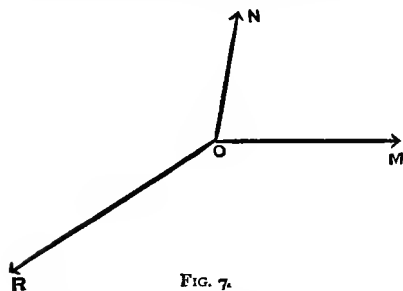


FIG. 7.

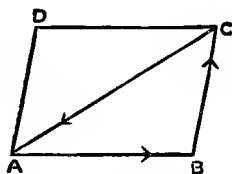


FIG. 8.

Let P, Q, R be the three forces acting on a particle at O , represented by OM, ON, OR , and let them be parallel and proportional to the sides of the triangle ABC taken in order; they will be in equilibrium.

Complete the parallelogram $ABCD$.

Then since AD is equal and parallel to BC ,
the forces P and Q will be represented by AB, AD .

But AC represents the resultant of forces represented by AB, AD ,

$\therefore AC$ represents the resultant of forces P and Q ;

that is, the resultant of P and Q is equal and opposite to the third force R ,

$\therefore$ the forces are in equilibrium.

Conversely: if three forces, acting on a particle, be in equi-

librium, they can be represented in magnitude and direction by the side of a triangle taken in order.

We will now prove a particular case of the above general proposition.

50. *If three forces, acting on a particle, be in equilibrium, and if a triangle be constructed having its sides parallel to the three forces, the sides of this triangle will be proportional respectively to those forces; that is to say, they will represent them in magnitude as well as in direction.*

This is called the **Converse of the Triangle of Forces**.

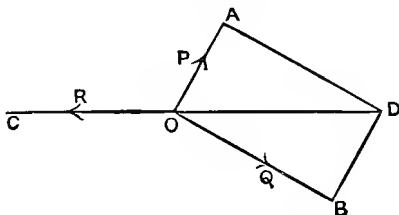


FIG. 9.



FIG. 10.

Let OA , OB , OC represent the three forces P , Q , R in equilibrium.

Let XYZ be the triangle, having its sides parallel to the forces P , Q , R respectively.

Complete the parallelogram $OADB$.

Then OD represents the resultant of P and Q .

But R is equal and opposite to the resultant of P and Q ,

$\therefore COD$ is a straight line, and $DO = OC = R$.

Now XY is $\parallel$ to OA , $YZ \parallel$ to AD , and $ZX \parallel$ to DO ,

$$\therefore XY : YZ : ZX :: OA : AD : DO$$

$$:: P : Q : R.$$

Corollary.—If two forces acting at a point be represented by two sides of a triangle taken in order, their resultant will be represented by the third side taken the opposite way.

51. If three forces acting on a particle are in equilibrium, each force is proportional to the sine of the angle between the other two.

Let P, Q, R be the three forces, α, β, γ the angles between them.

Draw the triangle $A B C$ with its sides taken in order parallel to $P Q, R$.

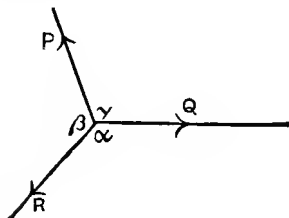


FIG. 11.

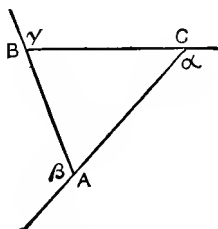


FIG. 12.

Then the exterior angle at $C = \alpha$

the „ „ $A = \beta$

„ „ „ $B = \gamma$.

Then $P : Q : R :: AB : BC : CA$ (Art. 50)

$:: \sin C : \sin A : \sin B$

$:: \sin \alpha : \sin \beta : \sin \gamma$.

$$\therefore \frac{P}{\sin \alpha} = \frac{Q}{\sin \beta} = \frac{R}{\sin \gamma}.$$

This is sometimes written :

$$\frac{P}{\sin (QR)} = \frac{Q}{\sin (RP)} = \frac{R}{\sin (PQ)}$$

where $\sin (QR)$ stands for the sine of the angle between Q and R , and so on.

This is called **Lami's Theorem**.

Polygon of Forces.

52. If several forces acting on a particle can be represented in magnitude and direction by the sides of a polygon taken in order, they will be in equilibrium.

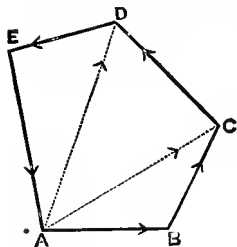


FIG. 13.

Let the several forces be represented in magnitude and direction by the sides AB , BC , CD , DE , EA .

Join A to all the angular points. Then by the triangle of forces, forces represented by AB , BC , CA are in equilibrium.

$\therefore AC$ represents the resultant of forces AB , BC .

So AD represents the resultant of forces AC , CD
i.e. of AB , BC , CD .

So also AE represents the resultant of forces AD , DE
i.e. of AB , BC , CD , DE .

Lastly, forces represented by AE , EA are in equilibrium.

Corollary.—From the above proof it is obvious that if several forces acting at a point be represented in magnitude and direction by the sides of an incomplete polygon taken in order, their resultant will be represented by the side which completes the polygon, taken from the extremity of the first side to the extremity of the last side; and if these two points coincide the resultant will be zero.

Resolution of Forces.

53. We have seen that if OA , OB represent two forces acting on a particle, and if the parallelogram $OACB$ be completed, the diagonal OC will represent the resultant force in magnitude and direction.

Hence, the force represented by OC can be substituted for the other two or *vice versa*; that is to say, the force represented

by OC can be resolved into two other forces by describing on OC as diagonal a parallelogram $OACB$; the forces represented by OA , OB will be components of the force represented by OC .

Now the number of parallelograms which can be constructed on any line as diagonal is infinite; hence there are an infinite number of ways of resolving any given force into two component forces, provided that there are no restrictions as to the magnitude or direction of the two components.

Example.—To resolve a given force F into two component forces of which one is given in magnitude equal to P , and the other has to make an angle θ with the direction of F .

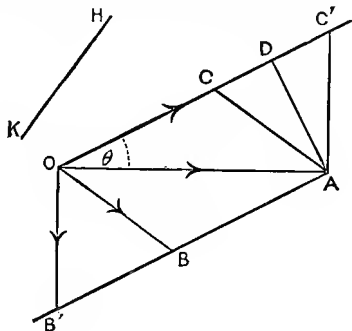


FIG. 14.

Let OA , HK represent the magnitude of the forces F , P respectively. Draw OC , making an angle θ with OA , and through A draw AB parallel to OC . With O and A as centres and radius equal to HK describe circles cutting AB and OC respectively in B , B' and C , C' .

Then $OBA C$, $OB' A C'$ are parallelograms, having OA for diagonal, and such that $OB = OB' = HK = P$, and $OC C'$ makes an angle θ with OA .

Hence there are generally two ways of resolving a given force into two components which satisfy the above conditions.

If AD be drawn perpendicular to OC , $AD = OA \sin \theta$. Hence, if $P = F \sin \theta$, the circles described with O and A as centres touch AB , OC respectively, and we only have one parallelogram formed, which is rectangular. In this case, therefore, there is only

one method of resolving the given force, and the other component is $F \cos \theta$.

If $P < F \sin \theta$, the circles will not meet O B, A C, and the question has no solution.

If $P > F$, $H K > O A$, and C, C' fall on opposite sides of O. We still have two parallelograms formed, but only one of them satisfies the required conditions; for if the angle $C O A = \theta$, the angle $C' O A$ must be $= 180^\circ - \theta$.

Examples.

1. Resolve a given force into two components, making given angles with it. Find the ratio of these components if the angles are 30° and 45° . *Ans.* $1 : \sqrt{2}$.

2. Resolve a given force into two others whose magnitudes are given. What conditions are necessary as to the given magnitudes of the components?

3. Resolve a force of 12 poundals into two others, one of which is 6 poundals and the other makes an angle of 30° with the force 12.

4. In Art. 53, Ex., if $P = F \sin \theta$, show that the other component $= \sqrt{F^2 - P^2}$.

5. If a force 4 be resolved into forces 3 and 2, find the cosines of the angles which the components make with the given force.

Ans. $\frac{7}{8}$ and $\frac{11}{8}$.

6. Resolve a given force into two components, one of which is given in magnitude and direction. What is the magnitude of the other component? *Ans.* $Q^2 = F^2 + P^2 - 2 F \cdot P \cdot \cos \theta$.

7. If a force equal to the weight of 5 lbs. be resolved into two others, one of which is equal to the weight of 4 lbs., making an angle of 60° with it, find the magnitude of the other component. If ϕ be the angle which it makes with the given force, show that $\sin \phi = \frac{2}{\sqrt{7}}$.

54. To find the resolved part of a force P in a direction making an angle θ with the given force.

Since a force can produce no change of position in a direction at right angles to itself, we must resolve the given force

into two components—one in the given direction, and the other at right angles to it.

Let OC represent the given force P , and DE the given direction.

Draw OX parallel to DE , making an angle θ with OC , and draw OY at right angles to OX .

Through C draw CA , CB , at right angles to OX , OY respectively.

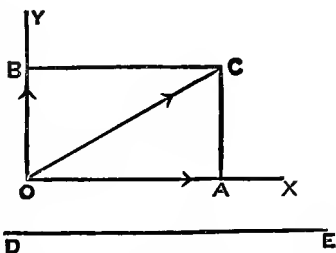


FIG. 15.

Then $OACB$ is a parallelogram, and therefore OA , OB represent two components of P .

Further, since OB is $\perp^r$ to OX , it produces no displacement in the direction OX ; therefore OA represents the total force in the direction OX .

$$\begin{aligned}\text{Now } OA &= OC \cos \theta = P \cos \theta, \\ \text{and } OB &= OC \sin \theta = P \sin \theta.\end{aligned}$$

Hence the resolved part of a force P in a direction making an angle θ with it is equal to $P \cos \theta$.

The other component $\perp^r$ to this one is equal to $P \sin \theta$.

55. *To find the resultant of any number of forces acting at a point in the same plane, their directions and magnitudes being given.*

This can be done by applying the parallelogram of forces as many times as is required.

Thus, first find the resultant of two of the forces, then find the resultant of this force and a third, and so on.

This process will be found extremely tedious, except in the most simple cases.

Again, the resultant might be found by the method of Art. 52.

This method is well adapted to geometrical measurements depending on accurate drawing, but the determination of the length and position of the line AE by trigonometrical calculation is very troublesome.

The following is the method which we almost invariably adopt :—

Resolve all the forces along two lines at right angles to each other.

We shall thus get two equivalent forces at right angles to each other, one of which is the sum of all the components in one direction, and the other the sum of all the components in a direction at right angles to it. We then find the magnitude and direction of the resultant of these two forces.

Thus, let $P_1, P_2, P_3, \&c.,$ be the forces acting on a particle at O , making angles $\theta_1, \theta_2, \theta_3, \&c.,$ respectively with a given line Ox .

Draw Oy perpendicular to Ox .

Resolve P_1 along Ox and Oy .

The resolved part of P_1 along $Ox = P_1 \cos \theta_1$ (Art. 54)

„ „ „ $Oy = P_1 \sin \theta_1$.

Proceed in a similar manner with all the other forces.

Let X be the algebraical sum of all the components along Ox .

„ Y „ „ „ „ „ Oy .

$X = P_1 \cos \theta_1 + P_2 \cos \theta_2 + P_3 \cos \theta_3 + ...$

$Y = P_1 \sin \theta_1 + P_2 \sin \theta_2 + P_3 \sin \theta_3 + ...$

The directions of X and Y will be given by their signs.

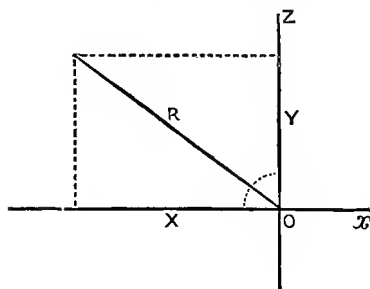


FIG. 16.

Hence, if R be the resultant force and ϕ the angle it makes with Ox ,

$$R^2 = X^2 + Y^2$$

$$\tan \phi = \frac{Y}{X}.$$

Corollary.—If the resultant force be zero,

$$R = 0$$

$$\therefore X^2 + Y^2 = 0$$

$$\therefore X = 0 \text{ and } Y = 0.$$

We thus get the following condition that any number of forces acting at a point should be in equilibrium.

The sum of the components of the forces resolved in any two directions at right angles to each other must each be zero.

Example.—Find the magnitude and direction of the resultant of forces 1, 2, 3, 4, 5, 6, 7, each making an angle of 30° with the next one.

Let Ox be the direction of force 1.

Draw $Oy \perp$ to Ox .

Let X and Y be the sum of all the components in the directions Ox , Oy respectively, R their resultant.

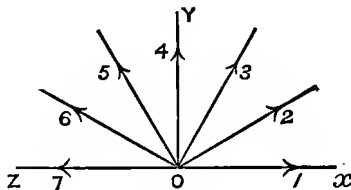


FIG. 17.

$$X = 1 + 2 \cos 30^\circ + 3 \cos 60^\circ + 4 \cos 90^\circ + 5 \cos 120^\circ \\ + 6 \cos 150^\circ + 7 \cos 180^\circ$$

$$= 1 + 2 \cdot \frac{\sqrt{3}}{2} + 3 \cdot \frac{1}{2} + 5 \left(-\frac{1}{2} \right) + 6 \left(-\frac{\sqrt{3}}{2} \right) + 7(-1)$$

$$= 1 + \frac{3}{2} - \frac{5}{2} - 7 + \sqrt{3} - 3\sqrt{3}$$

$$= -7 - 2\sqrt{3}$$

$$Y = 2 \sin 30^\circ + 3 \sin 60^\circ + 4 \sin 90^\circ + 5 \sin 120^\circ + 6 \sin 150^\circ$$

$$= 2 \cdot \frac{1}{2} + 3 \cdot \frac{\sqrt{3}}{2} + 4 + 5 \cdot \frac{\sqrt{3}}{2} + 6 \cdot \frac{1}{2}$$

$$= 1 + 4 + 3 + 3 \frac{\sqrt{3}}{2} + \frac{5\sqrt{3}}{2}$$

$$= 8 + 4\sqrt{3}$$

$$R^2 = (-7 - 2\sqrt{3})^2 + (8 + 4\sqrt{3})^2$$

$$= 173 + 92\sqrt{3}$$

$$R = \sqrt{173 + 92\sqrt{3}}$$

$$= 18.2 \text{ approx.}$$

If θ be the angle this resultant makes with Ox ,

$$\begin{aligned}\tan \theta &= -\frac{8 + 4\sqrt{3}}{7 + 2\sqrt{3}} \\ &= -\frac{32 + 12\sqrt{3}}{37} \\ &= -1.427.\end{aligned}$$

This lies between -1 and $-\sqrt{3}$, and therefore θ lies between 120° and 135° .

Its actual value is about 125° .

Example.—To explain by means of the *resolution of forces* how a sailing vessel can be made to go from one point to another against the wind.

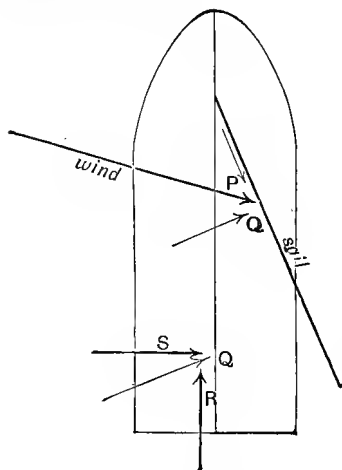


FIG. 18.

This is done by the process called *tacking*; that is, taking a zigzag course in directions making acute angles with the direction of the wind. In such a case the sail is placed so that its plane lies between the direction of the wind and the ship's course.

Suppose that F is the force of the wind on a square inch of sail. Resolve F into two components, P and Q respectively parallel and normal to the plane of the sail; of these, P has no effect on the sail, therefore Q represents the total *available* effect of the wind on each square inch of sail.

Resolve Q into its two components R and S respectively parallel and normal to the proposed course of the ship; of these R is entirely available to move the ship in the required direction. Although the component S is very often larger than R , it only has a small effect on the motion of the ship, since the shape of all ships is such that they move easily in the direction of their length

and only with difficulty in a direction perpendicular to their length.

The effect of *S* is called *leeway*, and is largely counteracted by the effect of the rudder, though there will always be a little leeway.

Thus we see how the total effect of the wind on the whole sail can be so resolved as to make the ship move in the direction of its length at an acute angle to the direction of the wind.

So, in the case of a kite, we have to consider the equilibrium of the weight of the kite, the tension of the string, and the force of the wind *perpendicular* to the plane of the kite.

56. Since velocities and accelerations are compounded by the Parallelogram Law in the same manner as Forces are, and since the results of Arts. 46–55 have all been derived from the Parallelogram of Forces, it follows that we shall have analogous propositions for velocities and accelerations. In fact, all the propositions contained in the above articles will hold *mutatis mutandis* for velocities and accelerations.

We shall thus have the Triangle and Polygon of Velocities and Accelerations, and we shall resolve and compound them in the same manner as we have resolved and compounded forces.

Examples.

1. What forces will just balance a system of four forces of 6, 7, 8, and 9, two of which act in one direction and the other two in the same straight line in the opposite direction?

2. A heavy uniform chain has 10 lbs. and 16 lbs. attached to its ends, and hangs in equilibrium over a smooth peg. If the greatest tension of the chain be 20 lbs. weight, find the weight of the chain.

3. In the above question, if the lengths of the chain on either side of the peg be as 3 : 2, find the weight of the chain.

4. Forces 3, 4, and 5 act in the same straight line. What is the greatest and least resultant they can have? How must they act if they are in equilibrium?

5. Forces 99 and 20 act on a point at right angles to each other. Determine the magnitude of their resultant.

6. Forces 1 and 2 act on a particle at an angle of 135° . Find the resultant.

7. Forces 5 and 4 act on a particle at an angle $\cos^{-1}\left(\frac{1}{5}\right)$. Find the resultant.

8. Forces 3 and 4 act at an angle of 60° . Find the magnitude of the resultant and the sines of the angles it makes with the components.

9. Find the resultant of forces 13 and 14 acting at an obtuse angle whose sine is $\frac{12}{13}$.

10. Forces 1, 2, and $\sqrt{3}$ acting at a point are in equilibrium. What are the angles between their directions?

11. A 10-lb. weight is supported by two forces, one of which acts horizontally and the other at an angle of 30° with the horizon. Find the magnitude of these forces.

12. Forces P and 2 P act at a point at an angle of 60° with each other. Find the magnitude and direction of the resultant.

13. Forces 7 and 8 act at an angle of 60° . Determine their resultants.

14. If the angle at which two forces act be increased, show that their resultant will be diminished.

15. Forces of $2\frac{1}{2}$, 6, $6\frac{1}{2}$ act on a particle and keep it at rest. Show that two of them are at right angles, and find the cosine of the angle between the greatest and least.

16. R and R^1 are the smallest and greatest forces which along with P and Q can keep a particle at rest. Show that if P, Q, $\sqrt{RR^1}$ are in equilibrium, two of these forces are perpendicular to each other.

17. A picture is suspended by a cord attached to its upper corners, and passing round a nail. Show that the longer the string the smaller will the tension be.

18. Two pictures of equal size and weight are suspended like the one in the preceding question. The strings are inclined at angles of 45° and 60° to the top of the pictures. Compare the tensions.

19. The resultant of two forces acting at a point is double of one of the forces P and at right angles to it. What must be the magnitude and direction of the other force?

20. A mass of 10 lbs. is suspended by a string and is acted on

by a horizontal force so that the string is inclined at an angle of 30° to the vertical. Find this force and the tension of the string.

21. Forces of 5 and 6 acting at a point have a resultant of 7. Find the angle between them.

22. Forces of 7 and 8 acting at a point have a resultant of 6. Find the angle between them.

23. Forces of 4 and 5 have a resultant of 7. Find the angle between them.

24. Forces of 5 and 6 have a resultant of 8. Find the angle between them.

25. Forces 2, P, and Q are in equilibrium; the angle between 2 and P is 135° and the angle between 2 and Q is 90° . Find P and Q.

26. Two forces act at an angle of 135° . Find the ratio between them in order that the resultant may be equal to the smaller.

27. P, and Q are two forces in the ratio of $2 : \sqrt{3}$; R is the resultant of P and Q when at right angles to each other; S is the resultant of R and Q. Find the magnitude and direction of S.

28. R is the resultant of two forces P and Q, S the resultant of R and P, and T the resultant of R and Q. Find the resultant of S and T.

29. Forces $P + Q$ and $P - Q$ act on a point at an angle whose cosine is $\frac{1}{2} \frac{P^2 + 2Q^2}{Q^2 - P^2}$. Find the force which will keep them in equilibrium. Show that P must be greater than 2 Q.

30. A B C is a triangle; D the middle point of B C. Show that the resultant of forces represented by A B, A C is represented by 2 A D.

31. Forces represented by lines drawn from any point O to the angular points of any triangle are equivalent to forces represented by the lines drawn from O to the middle points of the sides.

32. If forces act at the angular points of a triangle in directions towards and perpendicular to the opposite sides, and are proportional to those sides in magnitude, prove that they are in equilibrium.

33. Forces acting at the angular points of a triangle are represented in magnitude and direction by lines drawn from the angular

points to the middle points of the opposite sides. Prove that they are in equilibrium.

34. Three equal forces (P) act at a point parallel to the sides of a triangle, these sides being in the ratio of $3 : 4 : 5$. Find the magnitude of their resultant.

35. Two forces are represented in magnitude and direction by two chords AP , AQ of a circle, which are at right angles to each other. Find the magnitude and direction of their resultant.

36. Forces are represented by lines OA , OB , OC , OD drawn from any point O to the fixed points A , B , C , and D . Show that their resultant is $4 OG$, G being a fixed point.

37. R is the resultant of two forces P and Q acting at a point at right angles to each other; S is the resultant of R and P : if $Q = 2P$ prove that $S = 2P\sqrt{2}$.

38. In the above question if $Q = 3P$ prove that $S = P\sqrt{13}$.

39. Two forces acting at a point are represented by a side and diagonal of a square, the length of the side being a . Show that their resultant is $a\sqrt{5}$ and find its direction.

40. Forces P , Q , and R act at a point parallel to the sides of an equilateral triangle taken in order. Show that their resultant is $\sqrt{P^2 + Q^2 + R^2 - QR - RP - PQ}$.

41. Forces P , Q , R , and S act at a point parallel to sides of a square taken in order. Find the magnitude of the resultant.

42. $ABCD$ is a rectangle, E is the middle point of AB ; forces represented by AD , DE , EC , CB act at a point. Find their resultant.

43. $ABCD$ is a parallelogram; E and F are the middle points of AB and BC ; forces acting at a point are represented by AF , FB , AE , ED . Determine their resultant.

44. $ABCDEF$ is a regular hexagon and forces act at the point A in the directions AC , AF , DA , and are in equilibrium. The force along AF is 2 lbs. wt.; find the other two.

45. Forces 1, 2, 3 act at a point parallel to the sides of an equilateral triangle taken in order. Show that their resultant is $\sqrt{3}$.

46. Forces 1, 2, 3, 4, 5, 6 act from the centre of a regular hexagon to its corners. Find their resultant.

47. Forces 5, 13 and 20 act on a particle and are equally inclined to each other. Find their resultant.

48. The sides of a triangle taken in order are 3, 4, and 5 inches. Forces of 12, 16, 19 act along them. Find the resultant.

49. The sides of a quadrilateral are 1, 2, 9, 7 inches. Forces of 2, 4, 8, 14 act along them respectively. Find the resultant.

50. Forces 3, 5, and 6 act on a particle ; can they keep it at rest? If so, what is the angle between 6 and 5?

51. Forces $2P$, $\sqrt{3}P$ have a resultant P . Find the angle between them.

52. $ABCD$ is a quadrilateral, and forces acting at a point are represented in magnitude and direction by AB , BC , DC , AD . Find their resultant.

53. If the resultant of two forces be perpendicular to one of the components and make an angle of 30° with the other, find the ratio of the two components.

54. Resolve a force of 7 into two components, making angles of 30° and 60° with it.

55. Two forces acting at a point at right angles to each other are kept in equilibrium by a third force, making an angle of 30° with one of them. The greatest force is 3 lb. wt. ; what must the other two be?

56. A weight of 10 lbs. is suspended by a string 25 inches long. The weight is drawn aside until its vertical distance below the point is 20 inches. Find the least force which will keep it in this position.

57. $ABCDEF$ is a regular hexagon, and forces are represented by AB , AC , AD , AE , AF . Find their resultant.

58. Forces 1, 2, 3 act at the angular points of an isosceles right-angled triangle towards and perpendicular to the opposite sides, the last force being perpendicular to the hypotenuse. Find the resultant.

59. A quadrant is trisected, and forces 1, 2, 4, 8 act along the four radii to the points of section. Find the magnitude and direction of the resultant.

60. Forces 1 and 2 are at right angles to each other, and a force $3\sqrt{2}$ bisects the angle between them. Find the resultant.

61. Forces 10 and 3 are at right angles, and a force $2\sqrt{2}$ bisects this angle. Show that the resultant of the three is equal to the sum of the two first.

62. If any number of forces in one plane act at a point O, and a fixed line O A be drawn through that point, find an expression for the tangent of the angle which their resultant makes with O A.

63. If a given force be resolved into two equal forces, prove that the extremities of the lines representing these equal forces always lie on a fixed straight line.

64. If a given force be resolved into two components at right angles to each other, prove that the extremities of the lines representing those components always lie on the circumference of a fixed circle.

65. If a force $8P$ be resolved into two forces, each equal to $5P$, find the sine of the angle between the equal components.

66. Give a geometrical construction for resolving a force into two others inclined at a given angle, one of which is to be of given magnitude. Under what conditions would you get two values for the other component?

67. Resolve the force represented by the diagonal of a square into three equal forces, each being represented in magnitude by a side of the square, and one of them being coincident with a side of the square.

68. Explain the equilibrium of a kite in the air.

69. A string is attached to a kite at a point P, whose distance from the centre of gravity G is 5 feet. The resultant effective pressure of the air is perpendicular to the kite's surface and acts at a point R in PG such that PR equals 18 inches. The plane of the kite is inclined at an angle of 45° to the horizon. Find the tension of the string if the kite be 4 lb. wt.

70. A B C D is a parallelogram, P any point in its plane; through P lines are drawn parallel to the sides, cutting A B, C D in L, N respectively, and B C, A D in M, O respectively. Show that forces represented by M L, N O, A C if acting at a point would be in equilibrium.

Through O draw O C parallel to the direction of P and Q, and also a line parallel to A B the direction of F.

Suppose R_1 the resultant of P and F to act at O (Art. 9), and at O replace R_1 by its original components P and F acting parallel to their former directions ;

Similarly let R_2 acting at O be replaced by its original components Q and F acting parallel to their former directions.

We then have four forces acting at O, of which P and Q act along O C, and the two equal and opposite forces F act parallel to A B. These two forces F may be removed without disturbing the effect of the system ; and we are left with two forces P and Q acting along O C, and producing exactly the same effect as P and Q acting at A and B respectively.

Therefore if R be the resultant of P and Q,

$$R = P + Q ;$$

and it acts parallel to the direction of P and Q. Since C is a point in its line of action we may suppose R to act at C.

To find the position of C.

Since P, F, R_1 are parallel to O C, C A, O A respectively, they are proportional to them. (Art. 50.)

$$\therefore \frac{P}{F} = \frac{O C}{C A} \quad (1)$$

$$\text{So also } \frac{Q}{F} = \frac{O C}{C B} \quad (2)$$

$$\text{Dividing (1) by (2) we have } \frac{P}{Q} = \frac{C B}{C A},$$

$$\text{or } P \cdot C A = Q \cdot C B.$$

58. *To find the magnitude and position of the resultant of two unlike parallel forces acting on a body.*

The proof of this is almost identical with that of the pre-

ceding article ; we will, however, go through it in detail on account of its importance.

Let P and Q be two unlike parallel forces acting on a body at points A and B respectively.

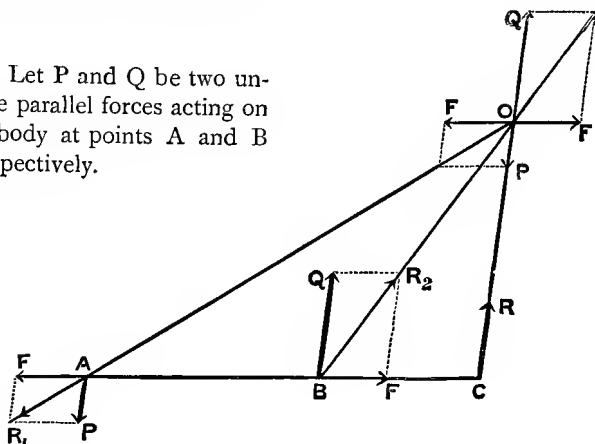


FIG. 20.

Suppose Q greater than P . Join AB and produce it both ways. At A and B apply two equal and opposite forces F in the line AB . This will not disturb the action of P and Q . The resultant of P and F acting at A can be found by the parallelogram of forces : let it be R_1 ; similarly let the resultant of Q and F be R_2 . Since the angle between P and F is equal to the angle between Q and F , but Q is greater than P , therefore the angle between R_2 and F is greater than the angle between R_1 and F (Art. 46, Cor.) ;

$\therefore$ the directions of R_1 and R_2 meet above AB (Euclid, Axiom 12).

Let them meet at O ;

through O draw OC parallel to the direction of P and Q , and also a line parallel to AB , the direction of F .

Suppose R_1 the resultant of P and F , to act at O , and at O replace R_1 by its original components P and F acting parallel to their former directions ; similarly let R_2 acting at O be replaced by its components Q and F acting parallel to their former directions. We now have four forces acting at O , of

which P and Q act along O C in opposite directions, and the two equal and opposite forces F act parallel to A B.

These two forces F may be removed without disturbing the effect of the system, and we are left with the two forces P and Q acting along O C in opposite directions and producing exactly the same effect as P and Q acting at A and B.

Therefore if R be the resultant of P and Q,

$$\begin{aligned} R &= Q - P \text{ in the direction } C O, \\ &= P - Q \text{ in the direction } O C, \end{aligned}$$

and it acts parallel to the direction of P and Q. Since C is a point in its line of action we may suppose R to act at C.

To find the Position of C.

Since P, F, R_1 are parallel to O C, C A, D A respectively, they are also proportional to them (Art. 50),

$$\therefore \frac{P}{F} = \frac{O C}{C A} \quad (1),$$

$$\text{So also } \frac{Q}{F} = \frac{C O}{B C} \quad (2).$$

$$\text{Dividing (1) by (2) we have } \frac{P}{Q} = \frac{B C}{C A},$$

$$\text{or } P \cdot C A = Q \cdot C B.$$

Note 1.—The result of this article is identical with that of the previous article.

The resultant of the unlike parallel forces P and Q is the algebraical sum of P and Q, and the distances of C from A and B have the same ratio in both cases.

Note 2.—We have supposed $Q > P$.

If P had been $> Q$, O, the point where R_1 and R_2 meet, would have been below A B and to the left of the line of action of P.

Hence, C would have been in A B to the left of A, such that $P \cdot C A = Q \cdot C B$.

59. The relation $P \cdot C A = Q \cdot C B$ enables us to find the distance of C from either A or B when the distance A B is

known and the magnitudes of the forces P and Q are also known.

Thus with the figure of Art. 58 put $CB = CA - AB$ in the equation.

$$P \cdot CA = Q \cdot CB$$

$$\therefore P \cdot CA = Q (CA - AB)$$

$$\therefore CA (Q - P) = Q \cdot AB$$

$$\therefore CA = \frac{Q \cdot AB}{Q - P}$$

$$\text{Similarly, } CB = \frac{P \cdot AB}{Q - P}$$

$$\text{When } P = Q$$

$$R = Q - P = 0$$

$$CA = \frac{Q \cdot AB}{Q - P} = \frac{Q \cdot AB}{0}$$

$$\therefore CA = \infty$$

$$\text{So also } CB = \infty.$$

Hence the point C is at an infinite distance from A and B.

The reason for this is apparent. For if $P = Q$ in the figure of Art. 58 the directions of R_1 and R_2 are parallel, and consequently the point O is removed to an infinite distance.

60. *Definition*.—If two equal parallel forces act on a body in opposite directions, but not in the same straight line, the system is called a **Couple**.

From the results at the end of the last article, *i.e.* $R = 0$ and $AC = BC = \infty$, we see that when a couple acts on a body it cannot be replaced by a single force at a finite distance. It may be looked upon as an infinitely small force acting at an infinitely great distance from the body.

The result of such a force cannot partake of the nature of a translation at all ; its effect on the body is that of a *twist* or *spin* only, about a point in the body.

We shall see in the next chapter how this effect may be measured.

61. To find the resultant of any number of parallel forces acting on a body in the same plane.

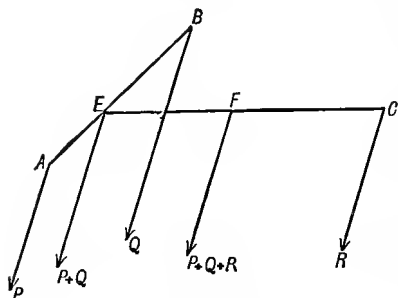


FIG. 21.

Let P , Q , R , &c. be parallel forces acting on a body at points A , B , C , &c. respectively.

The resultant of the parallel forces P and Q is a parallel force $P + Q$ which acts at a point E such that $P.AE = Q.BE$.

P and Q may now be replaced by this resultant acting at E . Join EC .

Then the resultant of the parallel forces $P + Q$ acting at E and R acting at C is the parallel force $P + Q + R$ acting at a point F such that $(P + Q).EF = R.FC$, and so on for any number of parallel forces.

Hence the resultant of any number of parallel forces is the algebraical sum of the forces, and acts at a point F , whose position depends only on the magnitude of the forces and the position of their points of application.

The position of F will not be altered if we suppose the lines of action of all the forces to be turned through the same angle at their points of application, provided that they remain parallel to each other.

On this account F , the point at which the resultant acts, is called the **Centre of the Parallel Forces**.

62. To find the position of the centre of a system of parallel forces whose points of application lie in a straight line.

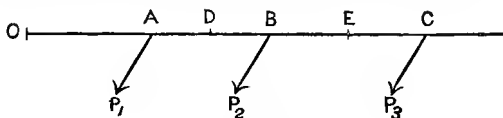


FIG. 22.

Let P_1 , P_2 , P_3 , &c. be parallel forces acting at points

A, B, C, &c. in a straight line, and let the distances of A, B, C &c. from a fixed point O in the line be x_1, x_2, x_3 , &c.

Let the resultant of P_1 and P_2 act at D.

$$\text{Then } P_1 \cdot AD = P_2 \cdot BD$$

$$P_1 (OD - x_1) = P_2 (x_2 - OD),$$

$$\therefore OD (P_1 + P_2) = P_1 x_1 + P_2 x_2.$$

Similarly if the resultant of $P_1 + P_2$ at D and P_3 at C act at E,

$$(P_1 + P_2) DE = P_3 \cdot CE$$

$$(P_1 + P_2) (OE - OD) = P_3 (OC - OE),$$

$$\begin{aligned} \therefore OE (P_1 + P_2 + P_3) &= OD (P_1 + P_2) + P_3 \cdot OC \\ &= P_1 x_1 + P_2 x_2 + P_3 x_3, \end{aligned}$$

and so on.

Hence if $\bar{x}$ be the distance of the point at which the resultant R acts from O, we have

$$\bar{x} \cdot R = P_1 x_1 + P_2 x_2 + P_3 x_3 + \dots$$

62a. To find the position of the centre of a system of parallel forces.

Let P_1, P_2, P_3 , &c. be parallel forces acting at points A, B, C, &c. in the same plane.

Let OX, OY be any two lines at right angles to each other in the plane in which the forces act.

We will find the distances of the centre of the parallel forces from OX and OY in terms of the distances of the points A, B, C from these lines and the magnitudes of the forces P_1, P_2, P_3 , &c.

Let D be the centre of the parallel forces P_1 and P_2 , and let E be the centre of the parallel forces P_1, P_2, P_3 .

Draw Aa, Bb, Cc, Dd, Ee perpendicular to OX.

Draw BH, DK perpendicular to Dd, Aa respectively, the triangles ADK, DBH are similar.

Then, since D is the centre of the parallel forces P_1, P_2 ,

$$P_1 \cdot AD = P_2 \cdot BD$$

$$\therefore \frac{P_1}{P_2} = \frac{BD}{AD} = \frac{DH}{AK} = \frac{Dd - Bb}{Aa - Dd}$$

$$\therefore P_1 (Aa - Dd) = P_2 (Dd - Bb)$$

$$Dd (P_1 + P_2) = P_1 \cdot Aa + P_2 \cdot Bb \quad (1)$$

This gives us Dd , the distance of D from Ox .

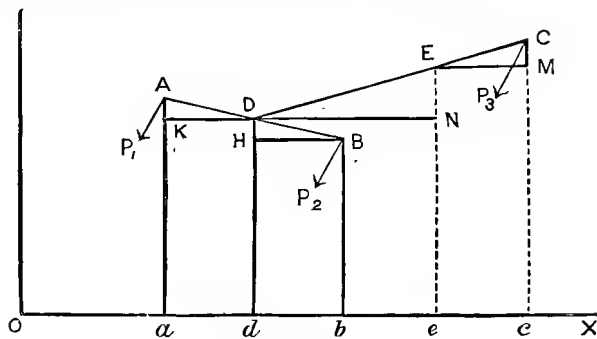


FIG. 23.

Now draw EM , $DN \perp$ to Cc , Ee respectively. The triangles DEN , ECM are similar, and since E is the centre of the parallel forces $P_1 + P_2$ and P_3

$$(P_1 + P_2) DE = P_3 \cdot CE$$

$$\therefore \frac{P_1 + P_2}{P_3} = \frac{CE}{DE} = \frac{CM}{EN} = \frac{Cc - Ee}{Ee - Dd}$$

$$\therefore (P_1 + P_2) (Ee - Dd) = P_3 (Cc - Ee)$$

$$\begin{aligned} \therefore Ee (P_1 + P_2 + P_3) &= (P_1 + P_2) Dd + P_3 \cdot Cc \\ &= P_1 \cdot Aa + P_2 \cdot Bb + P_3 \cdot Cc \text{ by (1),} \end{aligned}$$

and so on for any number of parallel forces.

Hence, if $P_1, P_2, P_3, \dots$ be any parallel forces acting at points $A, B, C, \&c.$, whose distances from a given straight line Ox are $y_1, y_2, y_3, \&c.$, then if $\bar{y}$ be the distance of the centre of the system from Ox we have

$$\begin{aligned} \bar{y} (P_1 + P_2 + P_3 + \dots) &= P_1 \cdot y_1 + P_2 \cdot y_2 + P_3 \cdot y_3 \\ &+ \dots \end{aligned}$$

Similarly if $x_1, x_2, x_3, \dots$ be the distances of A, B, C, &c. from the line O Y and $\bar{x}$ the distance of their centre, we shall have

$$\bar{x} (P_1 + P_2 + P_3 + \dots) = P_1 x_1 + P_2 x_2 + \dots$$

$$\begin{aligned} \text{Let } P_1 + P_2 + P_3 + \dots &= \Sigma (P) \\ P_1 x_1 + P_2 x_2 + P_3 x_3 + \dots &= \Sigma (P x) \\ P_1 y_1 + P_2 y_2 + P_3 y_3 + \dots &= \Sigma (P y). \end{aligned}$$

$$\text{Then } \bar{x} = \frac{\Sigma (P x)}{\Sigma (P)}$$

$$\bar{y} = \frac{\Sigma (P y)}{\Sigma (P)}.$$

Note 1.—The results of the preceding articles hold whether the parallel forces are like or unlike, provided always we remember to give the proper sign to the value of the force.

Note 2.—It is not necessary that the points A, B, C, &c. should all be on the same side of the lines O X, O Y.

The formulæ at the end of the article will hold, in whatever quadrant A, B, C . . . lie, provided that we give the proper algebraical signs to the coordinates x_1, x_2, x_3 , &c., and y_1, y_2, y_3 , &c.

Note 3.—If the points of application of the forces lie in the same straight line, take O, a fixed point in the straight line. Thus, if x_1, x_2, x_3 , &c. be the distances of the points of application of P_1, P_2, P_3 , &c. from O, and $\bar{x}$ be the distance of their centre from O,

$$\bar{x} = \frac{\Sigma (P \cdot x)}{\Sigma (P)}.$$

Example.—Parallel forces 1, 2, 3, 4, 5, 6 act at points in a straight line whose distances are 2 feet apart; the forces 1, 3, 5 act in one direction and the forces 2, 4, 6 in the other. Find the distance of their centre from a point half way between the points where the forces 3 and 4 act.

Let O be the point halfway between the points where the forces 3 and 4 act.

Then the distances of the points of application of the forces 1, -2, 3, -4, 5, -6 from the point O are 5, 3, 1, -1, -3, -5 feet respectively.

Hence, if $\bar{x}$ be the distance of their centre from O we have

$$\begin{aligned}\bar{x} &= \frac{\Sigma (Px)}{\Sigma (P)} \\ &= \frac{1 \cdot 5 + (-2) \cdot 3 + 3 \cdot 1 + (-4) \cdot (-1) + 5 \cdot (-3) + (-6) \cdot (-5)}{1 - 2 + 3 - 4 + 5 - 6} \\ &= \frac{21}{-3} = -7 ;\end{aligned}$$

that is, the resultant force is 3 in the direction of the forces 2, 4 and 6 acting at a point 7 feet to the left of O.

63. *To find the resultant of any forces acting on a body in the same plane.*

We first take any two of the forces, which do not form a couple, and find their resultant by the method of Arts. 46 or 57.

This reduces the number of forces by one.

We then combine this resultant with a third force, which does not form a couple with it, and find their resultant ; and so on.

Proceeding in this way we shall eventually be left with two forces only.

These two forces may be,

- (1) equal and opposite and collinear,
- or (2) unlike, parallel and equal, but not collinear,
- or (3) unlike, parallel and unequal,
- or (4) like, parallel and equal, *or* unequal,
- or (5) they may meet in a point, not being equal, and collinear.

In Case (1) the resultant is zero and the forces are in equilibrium ;

in Case (2) the resultant is a couple ;

in Cases (3), (4), and (5) the resultant is a single force.

Hence, if any number of forces act on a body in one plane, and are not in equilibrium, their resultant must be a single force or a couple.

Examples.

1. Like parallel forces 3 and 7 act at points A and B 20 inches apart. Where does the resultant act?
2. Parallel forces 4 and 9 act in opposite directions at points 15 inches apart. Where does the resultant act?
3. The resultant of like parallel forces 7 and 5 acts at a point $2\frac{1}{3}$ ft. from the lesser one. Find the distance between the forces.
4. The resultant of unlike parallel forces 9 and 7 acts at a distance of $4\frac{1}{2}$ ft. from the lesser one. Find the distance between them.
5. The resultant of two like parallel forces is $12\frac{1}{2}$ and acts at a point 2 feet from one force and 3 feet from the other. Find the forces.
6. Parallel forces 3 and 5 act in opposite directions at A and B, and their resultant acts 3 feet from A. Find A B.
7. The resultant of two unlike parallel forces is 2 and acts at a point 6 inches from one and 8 inches from the other. Find the forces.
8. A line is divided into four parts in the ratio of 1 : 3 : 5 : 7, and five equal parallel forces act in the same direction at the extremities of the line and at the points of section. Show that the point at which the resultant acts will divide the line in the ratio of 3 : 5.
9. Masses of 1 lb., 7 lb., 5 lb. and 3 lb. are placed 1 foot apart on a rod without weight. Find the point on which the rod will balance.
10. Particles of 2, 3, 4, and 5 lbs. are placed along a weightless rod at distances 2, 3, 4, and 5 feet from one end. Find the distance from that end of the point about which the rod will balance.
11. A beam 15 feet long and 300 lbs. weight is supported in a horizontal position by props at the ends. It will balance on a point 7 feet from one end. Find the pressure on the prop at that end.
12. A uniform rod O A, 12 inches long, is suspended by two vertical strings at its ends, which break when the tension is more

than 7 lbs. wt. Weights of 2 lbs. and 7 lbs. are attached to the rod at distances of 1 inch from O and 2 inches from A respectively. Find the greatest weight the rod can have.

13. Two like parallel forces of 10 lbs. wt. and 11 lbs. wt. act on a body, the distance between their lines of action being 1 foot. A force of x lbs. wt. is added to the former and taken from the latter, in consequence of which the resultant is moved 3 inches. Find x .

14. If the forces in the above question be unlike, and x lbs. be added to the 11 lbs., and taken from the 10 lbs., in consequence of which the resultant is moved 3 feet. Find x .

15. A tricycle (5 st. 4 lbs.) has the small wheel placed symmetrically 3 feet behind the two large wheels, which are 3 feet apart. The centre of gravity of the machine is 9 in. behind the front wheels and that of the rider (9 st.) 3 in. behind. Find the pressures on the ground of the different wheels.

16. Weights of 1 lb., 4 lb., 3 lb., and 2 lb. are fastened to a straight rod without weight at distances from one end of 1, 2, 3, and 6 inches respectively. Find the point about which the rod will balance.

17. Two like parallel forces, P and Q, have a resultant R. If Q be changed to $\frac{P^2}{Q}$ show that the new resultant will occupy the same position as if the forces had been interchanged.

18. A weightless rod, A B, 1 foot in length, is kept horizontal by a vertical string, C D, attached to a point, C, in the rod, 4 in. from A, and also by two equal vertical strings attached to the extremities A and B, and to two weights of 5 lbs. and 3 lbs. on a horizontal plane below the rod. What is the greatest force that can be applied along C D without disturbing equilibrium?

19. Four forces in the ratio of 1 : 2 : 4 : 3 act along the sides A B, B C, D C, A D respectively, whose side is one inch. Prove that their resultant is parallel to a diagonal of the square and find where it cuts A B.

20. Like parallel forces act at the vertices of a triangle, each force being proportional to the opposite side. Show that the centre of the parallel forces is at the centre of the circle inscribed in the triangle.

21. A B is a uniform rod of length n inches, and weight $(n + 1)$

W. To the rod, weights $W, 2W, 3W, \dots, nW$, are attached at distances $1, 2, 3, \dots, n$ inches respectively from A. Find the distance from A of the point on which the rod will balance.

22. ABC is a triangle, and D is the middle point of AB . Find the position of the resultant of forces represented by two parallel lines DE, BF , which trisect AC at E , and F .

23. Forces $3, 4$, and 5 act towards the same parts through points A, B , and C . Show that if their resultant bisects AC in D , the line of action of the force through B will bisect AD .

24. ABC is a triangle right angled at A , whose sides are a, b, c ; AD is the perpendicular from A on BC ; like parallel forces P, Q , and R acting through A, B, C respectively have their centre at the middle point of AD . Find the ratio of the forces.

CHAPTER V

MOMENTS, COUPLES

64. Hitherto we have only considered the *translating* power of a force on a body ; we must now consider the *rotating* power also (Art. 5).

There are three elements which determine a force :—

- (1) Its direction,
- (2) Its line of action *or* position,
- (3) Its magnitude.

The translating power of the force depends on (1) and (3) only, but the rotating power depends on (2) also.

This **rotating power** is called the **Moment of the Force**.

Definition.—The moment of a force about a point is the power of the force to produce rotation about the point.

The measure of this power will obviously depend on two things :—

- (1) The magnitude of the force,
- (2) The distance of its line of action from the fixed point.

Let us consider one or two examples.

In opening or shutting a door which rotates on hinges, we apply force by pulling or pushing near to the handle, since the further it is applied from the hinge the greater is the rotating power.

Again, in turning the capstan, the sailor is provided with a long pole which he fits into a hole in the capstan ; he then applies his pressure at the extremity of this arm, and thereby the rotating power of his push is largely increased.

In the figure, suppose that xyz is the cross section of a capstan movable about an axis perpendicular to the plane of the paper and cutting this plane in O .

Let F be the force which a man can exert, and let it be applied at the extremity of the arm A .

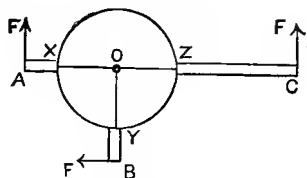


FIG. 24.

The further A is from the axis through O the greater is the rotating power of F about that axis.

Suppose that $OB = OA$, and $OC = 2 OA$, the rotating power of F acting at C is exactly double of its rotating power when acting at A or B , and we find that F acting at C will exactly balance, in rotating power, F acting at A , and F acting at B , or $2 F$ acting at A or B .

We thus see that if we double the distance of the line of action of the force from the fixed point we double the rotating power or moment about that point, or if we double the force and keep the distance the same, we shall also double the moment; if we double both force and distance the moment will be quadrupled.

The **distance** of the line of action of the force from the point or axis of rotation must always be taken the **shortest distance**; i.e. it is the length of the perpendicular from the point on the line of action of the force or between the axis of rotation and the line of action of the force.

Thus, if F be the force, D the perpendicular distance of its line of action from the point O or axis of rotation,

Moment of the force is proportional to F when D is constant, and is proportional to D when F is constant.

It is therefore proportional to the product $F \cdot D$, and may be taken equal to $F \cdot D$, provided we take our unit moment to be the rotating power of the unit force acting at unit distance; because



FIG. 25.

$$\frac{\text{Moment of force } F \text{ acting at distance } D}{\text{Moment of force } 1 \text{ acting at distance } 1} = \frac{F \cdot D}{1 \cdot 1}.$$

Thus, if we take 1 lb. wt. as our unit force and 1 foot as our unit distance, then the moment of a force of 4 lbs. wt. acting at a distance of 3 feet from a point or axis of rotation is 12 (units of moment).

The measure of the moment of a force F about a point or axis is $F \cdot D$, where D is the perpendicular distance of the line of action of the force from the point or axis of rotation.

The conception of the moment of a force and the method of measuring moments are of the greatest importance; the use of them in solving questions of equilibrium in Statics will occur over and over again. The student is advised to read again the previous article before passing on to the next.

65. *Geometrical representation of the moment of a force about a point.*

Let F be a force acting on a body at the point A .

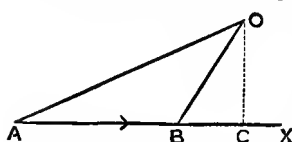


FIG. 26.

Let O be any point, and D the length of the perpendicular OC , drawn from O to AX , the line of action of F ; then the moment of F about O is $F \cdot D$.

Let AB represent the force F , so that AB contains F units of length; join OA , OB .

Then the area of the triangle OAB

$$= \frac{1}{2} \times \text{base} \times \text{altitude. (Euclid I. 41.)}$$

$$= \frac{1}{2} AB \cdot OC$$

$$= \frac{1}{2} F \cdot D. \text{ units of area.}$$

Hence the measure of the moment of F about O is numerically equal to twice the area of the triangle formed by joining O to the extremities of the line which represents the force F .

Note.—A B may be taken anywhere in the line of action of F without altering the area of the triangle O A B, because triangles on equal bases and of the same altitude are equal in area.

Corollary.—If the point O is in the line of action of F, O C (*i.e.* D) becomes = 0, hence the value of the moment of F about O becomes = zero ; that is, a force has no tendency to produce rotation about a point in its own line of action, or **a force has no moment about a point in its own line of action.**

In this case the area of the triangle O A B obviously becomes zero also.

Conversely : If the moment of a force about any point is zero, the point is in the line of action of the force.

66. This method of representing a moment geometrically only gives us a measure of the value of the moment ; it does not represent the moment completely, for it does not give us the direction of rotation.

If the direction of F were reversed, or if O were the same distance on the other side of A X, the moment of F would be reversed, but the area of the triangle O A B would remain the same. We therefore make the same convention with reference to direction of rotation as we make in trigonometry with reference to the direction of revolution and the measurement of angles. We consider moments which tend to turn the body in one direction as positive moments, and those which tend to turn the body in the opposite direction as negative moments.

It is usual to take the direction opposite to that of the hands of a watch as the positive direction of rotation, the reverse direction being negative. This convention is not necessary ; the student may take either direction of rotation as the positive one, provided that the reverse direction is taken as negative.

If a number of forces act on a body and the sum of all the moments about a point in one direction is equal to the sum of all the moments about the same point in the opposite direction, the body has no tendency to rotate about that point, and the algebraical sum of the moments about that point is zero.

67. We have defined the **Resultant** of two or more forces to be the **single force whose effect is equivalent to the combined effect of the forces**. We have shown how to find the Resultant of Forces in one plane as far as the *translating* power of the forces is concerned ; it is now necessary for us to prove that the force which is the resultant of other forces with reference to their *translating* power will also be their resultant with reference to their *rotating* power. This we shall first prove in the case of two forces which meet at a point or are parallel.

68. *If two forces meeting at a point act on a body, the algebraical sum of their moments about a point O in their plane is equal to the moment of their resultant about the same point.*

Case 1. When the lines of action of the forces are on the same side of O.

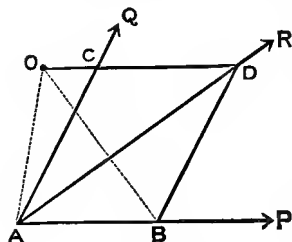


FIG. 27.

Case 2. When the lines of action of the forces are on opposite sides of O.

Let P, Q be two forces meeting at A, R their resultant ;

Let O be a point in the plane so that P and Q act on the same side of O.

Through O draw OCD parallel to the line of action of P, cutting the lines of action of Q and R in C and D ;

Through D draw DB $\parallel$ to AC.

Then ABDC is a parallelogram whose diagonal is AD, and AB, AC, AD are in the directions of P, Q, R respectively.

Therefore AB, AC, AD represent the forces P, Q, R completely.

Join OA, OB.

Then $\triangle OAB = \triangle DAB = \triangle CAD$,

$\therefore$ the algebraical sum of the moments P and Q about O

$$= 2 \triangle OAB + 2 \triangle OAC$$

$$= 2 \triangle CAD + 2 \triangle OAC$$

$$= 2 \triangle OAD$$

$$= \text{moment of R about O.}$$

Case 2. Let O lie between the line of action of P and Q .

With the same construction as in Case 1,

the algebraical sum of the moments of P and Q about O

$$\begin{aligned} &= 2 \triangle OAB - 2 \triangle OAC \\ &= 2 \triangle CAD - 2 \triangle OAC \\ &= 2 \triangle OAD \\ &= \text{moment of } R \text{ about } O. \end{aligned}$$

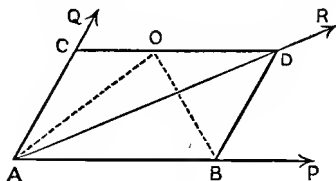


FIG. 28.

69. *If two parallel forces act on a body, the algebraical sum of their moments about a point O in their plane is equal to the moment of their resultant about the same point.*

Case 1. When the parallel forces are in the same direction,

Draw $OACB$ perpendicular to the lines of action of P , Q , R , cutting them in A , B , C respectively.

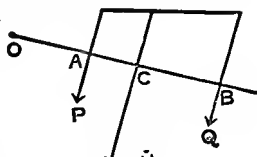


FIG. 29.

$$\text{Then } P \cdot AC = Q \cdot BC.$$

Sum of the moments of P and Q about O

$$\begin{aligned} &= P \cdot OA + Q \cdot OB \\ &= P(OC - AC) + Q(OC + CB) \\ &= (P + Q)OC - P \cdot AC + Q \cdot BC \\ &= R \cdot OC \quad \text{since } P \cdot AC = Q \cdot BC. \end{aligned}$$

Note.—We have here taken the positive direction of rotation the same as that of the hands of a watch.

Case 2. When the parallel forces act in opposite directions.

Suppose $Q > P$ so that $R = Q - P$ in the direction of Q .

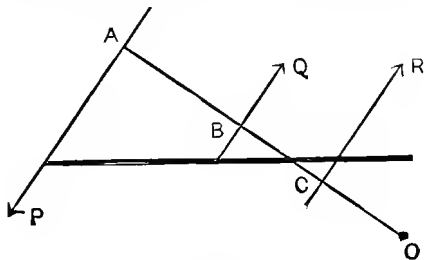


FIG. 30.

Sum of the moments of P and Q about O.

$$\begin{aligned}
 &= P \cdot OA - Q \cdot OB \\
 &= P(OC + CA) - Q(OC + CB) \\
 &= P \cdot OC - Q \cdot OC + P \cdot CA - Q \cdot CB \\
 &= (P - Q) OC \quad \text{since } P \cdot CA = Q \cdot CB \\
 &= -R \cdot OC \\
 &= \text{moment of R about O.}
 \end{aligned}$$

Note.—We have taken the positive direction of rotation opposite to that of the hands of a watch.

70. *The sum of the moments of any two forces in one plane about a point in the line of action of their resultant is zero.*

This follows at once from the result of the two previous articles and the Corollary of Art. 65.

It can be proved directly in any given case.

Conversely: *If the algebraical sum of the moments of any two forces about a point in their plane be zero, that point must lie in the line of action of their resultant.*

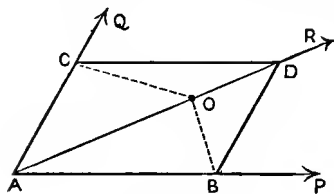


FIG. 31.

With the usual notation.

Since the sum of the moments of P and Q about O is zero,

$$\therefore \Delta OAB = \Delta OAC;$$

$\therefore$ since they are on the same base OA, they are of equal altitude.

i.e. B and C are equally distant from AO.

$\therefore$ AO is the direction of the diagonal of the parallelogram ABCD.

$\therefore$ O lies on the line of action of R.

71. The results of the preceding articles may now be extended to any number of forces in a plane which have a single force for their resultant.

For we have proved the theorem to be true for the resultant of two forces whether they meet in a point or are parallel. By

taking this resultant with a third force, as in Art. 63, we can prove the theorem true for three forces ; and so on for any number of forces which have a single force for their resultant.

We may therefore say :—

If any number of forces, which have a single force for their resultant, act on a body, the algebraical sum of their moments about a point in their plane is equal to the moment of their resultant about that point.

Whence it follows that

The algebraical sum of the moments of any number of forces in a plane about a point in the line of action of their resultant is zero.

And conversely :

If the sum of the moments of any number of forces in a plane which are not in equilibrium about a point in their plane be zero, that point must lie in the line of action of their resultant.

Note.—If the forces are in equilibrium their resultant is zero ; consequently the algebraical sum of their moments about **any** point in their plane must be zero ; since a system of forces in equilibrium is equivalent to no force at all.

To make this clearer, suppose that R is the resultant of any number of forces in one plane, and let p be the length of the perpendicular drawn from any point O in the plane on the line of action of R .

Then, sum of the moments of all the forces about O
 $=$ moment of R about O
 $= R \cdot p$.

If $R \cdot p = 0$,

$R = 0$, or $p = 0$;

if $R = 0$, the forces are in equilibrium,

if $p = 0$, the point O lies on the line of action of R .

72. In the preceding article we have supposed that the forces had a single force as their resultant.

We have seen in Art. 59 that in certain cases the resul-

tant of a number of coplanar forces is not a single force acting at a finite distance, but a couple ; that is, a system of two equal and parallel forces, acting in opposite directions, and not in the same straight line. The translating power of such a system is zero, because $R = 0$ (Art. 60) ; but its rotating power is never zero ; in fact, as we shall show in the next article, the rotating power of such a system about **any** point in its plane is always the same.

73. The algebraical sum of the moments of the two forces of a couple about any point in their plane is constant.

Let P, P be the forces of the couple acting at points A and B .

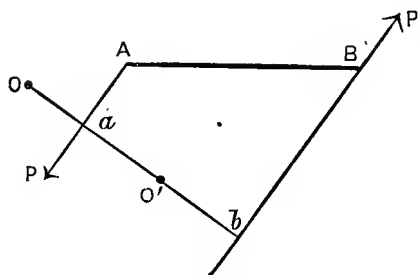


FIG. 32.

O any point in their plane.

Draw $Oa \perp$ to the lines of action of P, P .

The algebraical sum of the moments of P, P about O

$$\begin{aligned} &= P \cdot Ob - P \cdot Oa \\ &= P (Ob - Oa) \\ &= P \cdot ab. \end{aligned}$$

Similarly, if O were between the lines of action of the forces, as at O' ,

The algebraical sum of the moments about O'

$$\begin{aligned} &= P \cdot O'b + P \cdot O'a \\ &= P (O'b + O'a) \\ &= P \cdot ab. \end{aligned}$$

The length of ab , the perpendicular distance between the lines of action of the two forces, is the same at every point.

Let $ab = p$, p is called the **Arm** of the couple.

$P \cdot p$ is called the **Moment** of the couple. It is the product of either force into the arm, and is constant.

Note.—The result of this article is not inconsistent with the result of Art. 71, though at first sight it might be thought that since $R = 0$, the moment of R about a point in the plane should

also be = 0. It must, however, be remembered that though $R = 0$, yet its line of action is removed to an infinite distance (Art. 59), consequently its moment about any point O—which is = $R \times$ (its $\perp^r$ distance from O)—becomes $0 \times \infty$; and since the expression $0 \times \infty$ may have any finite value, there is no reason why it should not be equal to $P \cdot p$.

74. Since the translating power of every couple is zero, and the rotating power about *any* point in its plane is measured by its moment, it follows that the effect of a couple on a rigid body depends only upon **the magnitude of its moment**.

That is to say, the effect of a couple is not altered by moving the points of application of the equal forces or by turning the arm of the couple through any angle, provided we do not alter the forces or the length of the perpendicular distance between them.

Further, any couple can be replaced by another couple of equal moment in its own plane.

Thus, if it be required to find the arm of a couple Q, Q which will have the same effect as the couple P, P with an arm p , the only necessary condition is

Moment of couple Q, Q = moment of couple P, P.

Hence if q be the arm of the new couple,

$$Q \cdot q = P \cdot p$$

$$\therefore q = \frac{P \cdot p}{Q}.$$

Similarly, if the arm be given and the magnitude of the forces be required,

$$Q = \frac{P \cdot p}{q}.$$

75. *A couple may be transferred to any plane parallel to its own without altering its effect on a rigid body.*

This also is obviously true since the effect of a couple is solely a rotatory effect about some axis perpendicular to its plane; but the proof of the above by means of Art. 57 is of sufficient interest to authorise its insertion here.

Suppose the couple P, P to act with an arm A B, and to

be moved out of the plane of the paper into a parallel plane, keeping the arm AB parallel to itself.

Let ab be the new position of AB .

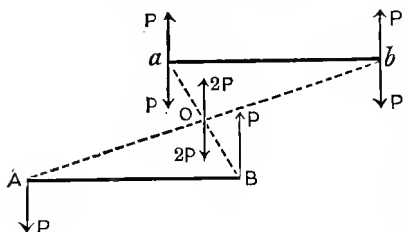


FIG. 33.

At both a and b apply the two equal and opposite forces P, P parallel to the forces of the couple. This will have no effect on the equilibrium of the system.

Join $A b, B a$. Then since AB, ab are equal and parallel, Ab, Ba bisect each other at O ; therefore, the resultant of the equal parallel forces P and P acting downwards at A and b is $2P$ acting downwards through O ; and the resultant of the forces P, P acting upwards at B and a is $2P$ acting upwards through O .

Now the equal and opposite forces $2P$ and $2P$ at the point O may be removed without disturbing the equilibrium of the system, and we are left with the couple P, P acting with the arm ab , producing exactly the same effect as it did with the arm AB .

Hence we see that a couple may be replaced by any other couple of equal moment acting in any position in its own plane, or in a plane parallel to its own, provided the direction of rotation remain the same.

76. To find the resultant of any number of couples acting in parallel planes.

Let $P, Q, R \dots$ &c. be the forces of the couples
 $p, q, r, \dots$ &c. their arms,

and let the planes in which the couples act be all parallel to the plane of the paper.

First, all the couples may be transferred to the plane of the paper without altering their effect.

They may then all be replaced by couples whose arms are equal to a . Thus, if $P' Q' R'$ be the forces of the new couples,

$$P' = \frac{P \cdot p}{a}; \quad Q' = \frac{Q \cdot q}{a}; \quad R' = \frac{R \cdot r}{a}, \text{ \&c.}$$

They may then be moved about in the plane until their arms are coincident with any line AB of length $= a$.

Then, if $X = P' + Q' + R' + \dots$ we have as the resultant of all the couples, a single couple $= X \cdot a$

$$\begin{aligned} &= (P' + Q' + R' + \dots) \times a \\ &= P' a + Q' a + R' a + \dots \\ &= P p + Q q + R r + \dots \end{aligned}$$

That is, the resultant of all the couples is a couple whose moment is equal to the algebraical sum of the moments of the original couples.

Note that in estimating the algebraical sum of the moments of the couples we choose either direction of rotation as the positive one, and then call those moments which are in the opposite direction 'negative.'

Thus, if the moments $P \cdot p$ and $Q \cdot q$ have rotations in opposite directions, they are of opposite signs. So also in the finding of the value of X , if $P \cdot p$ and $Q \cdot q$ be of opposite sign, P' and Q' at A will be in opposite directions, and likewise at B .

77. *To find the resultant of a couple and a force in the same plane.*

Let the moment of the couple be $P \cdot p$ and let the force be Q .

Let Q meet the direction of one of the forces of the couple $P \cdot p$ in A .

Let R be the resultant of P and Q , produce R to cut the direction of the other force of the couple in B .

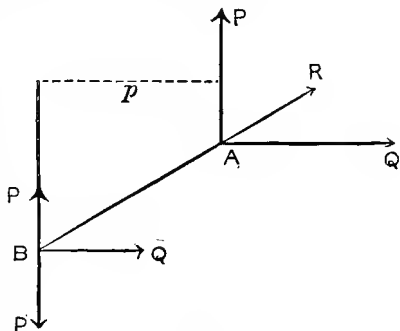


FIG. 34.

Suppose R to act at B and replace it by its original components P and Q acting parallel to their original directions.

Then the two equal forces P, P acting at B are in equilibrium and may be removed, and we are left with the single force Q acting at B parallel to the original direction of Q acting at A .

Hence the resultant of a single force and a couple is a single force equal and parallel to the original single force.

Or thus : replace the couple $P \cdot p$ by the couple $Q \cdot q$ of equal moments, and move the arm of this new couple until one end of it is at A , a point in the line of action of Q , and the arm is at right angles to AQ .

Then the equal and opposite forces Q, Q at the point A may be removed without disturbing the equilibrium of the system, and we have left the single force Q acting at C parallel to its original direction and producing the same effect

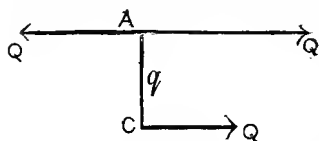


FIG. 35.

as the couple $P \cdot p$, and the force Q acting at A .

From this it is obvious that a single force and a couple can never produce equilibrium.

78. As an example of the results contained in this chapter let us find the resultant of a number of parallel forces acting on a rigid body in one plane.

Let the plane be the plane of the paper. Let O be a fixed point in the plane and draw Ox perpendicular to the direction of the parallel forces.

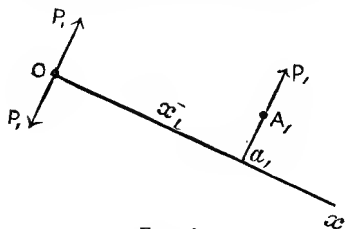


FIG. 36.

$Oa_2, Oa_3, \&c.$ be $x_1, x_2, x_3, \&c.$

At O apply two equal and opposite forces equal and parallel to P_1 .

Then P_1 acting at A_1 may be replaced by P_1 acting at O , and the couple whose moment is $P_1 \cdot x_1$.

Similarly all the other forces may be replaced by an equal force acting through O , and a couple.

$$\text{Let } P_1 + P_2 + P_3 + \dots = \Sigma(P)$$

$$\text{and } P_1 x_1 + P_2 x_2 + P_3 x_3 + \dots = \Sigma(Px).$$

Then the system of parallel forces P_1, P_2, P_3 , &c. may be replaced by $\Sigma(P)$ acting through O parallel to the directions of the forces, and the couple $\Sigma(Px)$.

To find the conditions of equilibrium of such a system.

It is obvious that there cannot be equilibrium if neither the single force $\Sigma(P)$ nor the couple $\Sigma(P \cdot x)$ vanish (Art. 77); neither will there be equilibrium if one of these vanishes and not the other; therefore they must both vanish;

$$i.e. \Sigma(P) = 0 \qquad \Sigma(P \cdot x) = 0$$

i.e. the algebraical sum of the forces must be equal to zero, and the algebraical sum of the moments of these forces about any point in their plane must also be zero.

Note.—This single force $\Sigma(P)$ and the couple $\Sigma(Px)$ can always be reduced to the single force $\Sigma(P)$ acting at a distance from O equal to $\frac{\Sigma(Px)}{\Sigma(P)}$ by the last article.

This result agrees with that of Art. 62.

79.* Similarly we may find the resultant of any number of forces in one plane.

Take any point O in the plane and draw any two lines Ox, Oy at right angles to each other.

Let P_1 be one of these forces acting at a point A_1 whose coordinates referred to Ox and Oy are (x_1, y_1) .

Let the components of P_1 parallel to Ox and Oy be X_1, Y_1 respectively, and

let a_1 be the angle P_1 makes with Ox_1 ,

$$\text{then } X_1 = P_1 \cos a_1 \text{ and } Y_1 = P_1 \sin a_1.$$

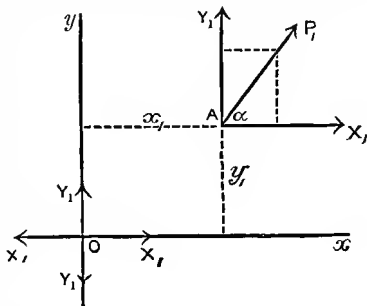


FIG. 37.

At O apply two equal and opposite forces X_1 parallel to Ox
 and " " Y_1 " Oy

Thus P_1 may be replaced by the forces X_1 Y_1 acting at O, the couple $Y_1 \cdot x_1$ from the axis of x to that of y in one direction and the couple $X_1 \cdot y_1$ from the axis of y to that of x in the opposite direction,—that is, the couple $Y_1 \cdot x_1 - X_1 \cdot y_1$ from the axis of x to the axis of y .

Proceeding in this manner with all the forces, proper attention being paid to the signs of the coordinates of the points of application of the forces and to the measurements of the angles $\alpha_1, \alpha_2, \alpha_3$, &c. in the same direction from the line Ox , we shall have as the final resultant, with the usual notation,

$$\Sigma(X) = \Sigma(P \cos \alpha) \parallel \text{to } Ox$$

$$\Sigma(Y) = \Sigma(P \sin \alpha) \parallel \text{to } Oy$$

and the couple $\Sigma(Yx - Xy)$ from the axis of x to that of y .

If R be the resultant of the single forces acting through O we have

$$R^2 = \{\Sigma(X)\}^2 + \{\Sigma(Y)\}^2$$

and if θ be the angle it makes with Ox

$$\tan \theta = \frac{\Sigma(Y)}{\Sigma(X)}.$$

As in the previous articles, if neither $R = 0$ nor $\Sigma(Yx - Xy) = 0$, the force and the couple may be reduced to a single force R parallel to this R and at a distance from O equal to $\frac{\Sigma(Yx - Xy)}{R}$.

For the equilibrium of the system we must have, as in the previous article,

$$R = 0 \text{ and } \Sigma(Yx - Xy) = 0$$

$$\text{or } \Sigma(X) = 0, \Sigma(Y) = 0 \text{ and } \Sigma(Yx - Xy) = 0.$$

We have seen in Arts. 68, 69 that the moment of the resultant P_1 about any point O is equal to the moments of its components X_1, Y_1 about the same point.

Hence if p_1 be the perpendicular distance of P_1 from O

$$\Sigma(Yx - Xy) = \Sigma(P \cdot p);$$

hence our conditions of equilibrium for any number of forces acting on a rigid body in one plane are :

The sums of the resolved parts of the forces parallel to any two straight lines at right angles to each other vanish, and the sum of the moments of these forces about a point in the plane must also vanish.

Example.—If the moments of two couples be equal in magnitude but opposite in direction, prove that the four forces which make up the couples are in equilibrium.

Let the couples be P, P at an arm p ,

and Q, Q " " q .

Let one force of each couple P and Q meet in A , and let the other force of each couple meet in B .

Draw AC, AD perpendicular to the directions of the forces P and Q , which meet at B .

Then since $P \cdot p = Q \cdot q$

$P \cdot AC = Q \cdot AD$

that is, the moments of P and Q acting at B about A are equal ;

$\therefore A$ is a point in the resultant of P and Q ;

$\therefore$ the resultant of P and Q acting at B is along BA .

Similarly the resultant of P and Q acting at A is along AB .

But the two resultants are obviously equal in magnitude, because the angle between P and Q at A is the same as it is at B .

Hence they are both equal in magnitude and opposite in direction, and are therefore in equilibrium.

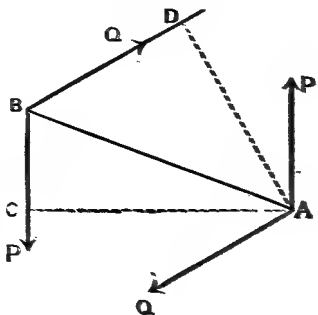


FIG. 38.

Examples.

1. Explain why a stick, which will support a large longitudinal tension, can be broken by a much smaller force at right angles to its length.

2. Compare the moments about a point of a force of 12 oz. wt. acting in a line whose distance from the point is 2 ft. 6 in., and of another of 5 oz. wt. acting at a distance of 3 ft.

3. A triangular lamina has a point within it fixed, and is acted on by forces completely represented by the sides of the triangle taken in order. Show that the moment of the force required to maintain equilibrium is represented by twice the area of the triangle.

4. Extend the above question to the case of a polygon.

5. What is the result if the fixed point be outside the lamina?

6. Masses of 7 lb., 1 lb., 3 lb., and 5 lb. are placed on a rod without weight 1 foot apart. Find the point on which the rod will balance.

7. The sides AB , BC of a rectangle $ABCD$ are respectively 3 ft. and 4 ft., and forces of 6, 8, 10 act along AB , BC , and CA respectively. Find the algebraical sum of their moments about A and D respectively.

8. The sides of a triangle are 9, 12, and 15 ft., and a force P acts along each of its sides. Find the moments about each of the angular points.

9. A straight rod AB without weight hinged at A is in equilibrium under the action of two forces P and Q in one plane; P acts at right angles to the rod at a point 2 in. from A ; Q acts at an angle 30° with the rod at a point 1 inch from A . Find the ratio of P to Q that the rod may have no tendency to turn about A .

10. AB , AD are two straight lines, and O a point outside the angle ABD ; BM is drawn parallel to OA , meeting AD in M . Prove that the moments about O of forces represented by AB , AM are equal.

11. ABC is a triangle, and OA is parallel to BC . Prove that the moments of forces represented by AB , AC about O are equal.

12. Show that four forces, represented in all respects by the sides of a quadrilateral, taken in order, are equivalent to two equal and unlike parallel forces, through the extremities of either diagonal, each being represented by a line equal and parallel to the other diagonal.

13. A circular disc suspended by its centre has strings from the centre hanging down over the edge, and supporting weights of 20 oz., 21 oz., and 29 oz.; the disc rests horizontally. Find the inclination to each other of the strings supporting the lighter weights.

14. A plank 5 ft. long and weighing 2 cwt. is supported at each end by a prop. Find at what point of the plank a weight of $2\frac{1}{2}$ cwt. must be placed in order that one prop may then support twice as much of the whole weight as the other.

15. A bar 16 in. long is balanced on a fulcrum at its middle. On the right arm are suspended 4 oz. and 3 oz. at distances of 5 in. and 7 in. respectively from the middle, and on the left arm 5 oz. at a distance of 5 in. from the middle and w at the end. Determine w .

16. A rod 6 in. long and 1 lb. wt. is supported by two vertical strings at its ends: a 3 lb. wt. is attached to the rod at a distance of 1 in. from one end. At what distance from the other end must a 4 lb. wt. be attached in order that the tensions of the two strings may be equal?

17. A heavy beam AB of known length and weight is movable in a vertical plane about a hinge at A , and is supported by a cord PQ of given length, P being fastened to the beam and Q to a point vertically above A . $AP = p$ and $AQ = q$. Prove that the tension of the cord will be the same when $AP = q$ and $AQ = p$.

18. $ABCD$ is a square whose side is a ; forces of magnitudes 2, 4, 2, 4 act along AB , BC , CD , DA respectively. Find their resultant.

CHAPTER VI

CENTRE OF GRAVITY

80. BEFORE proceeding to consider the general conditions of equilibrium of forces which keep a body at rest, we shall in the present chapter consider that force which is the most important of all forces, since it acts upon all bodies with which we are concerned ; *i.e.* the force which we call ‘gravity’ (Art. 7).

This force is due to the attraction of the earth for all matter within a finite distance of its surface, and the measure of this force acting on any particular body is called the ‘weight of the body.’

The measure of the force of attraction between any two *particles* is proportional to $\frac{m m'}{r^2}$ where m, m' are the masses of the particles and r the distance between them ; and the line of action of this force is along the line joining the particles.

We shall assume that the attraction due to the whole mass of the earth is the same as that due to a particle of equal mass placed at the earth’s centre. Hence the line of action of the force of gravity on any particular particle is the line joining the particle to the centre of the earth.

This is called the **vertical line** at the point.

Since the earth is not a perfect sphere, the force of gravity varies from point to point of the earth’s surface, being least at the equator and greatest at the poles ; because the distances of points on the equator from the earth’s centre are greater than the distances of other points on the earth’s surface from its centre.

Since the mass of the earth is constant, it follows that the weight of any particle at any particular spot on the surface of the earth varies as the mass of the particle only.

Hence, at the same place, equal masses have equal weights ; and at different places, the same mass has different weights.

Thus if w be the weight, m the mass of any particle,

$$\begin{aligned}\text{the ratio } \frac{w}{m} &= g \\ \text{or } w &= m \cdot g\end{aligned}$$

where g is always the same at the same place, but varies at different places on the earth's surface.

81. Now any body may be considered as an agglomeration of particles, and the weight of any body is the resultant of the weights of all the particles of which the body is composed. All these weights act in vertical lines, that is, in lines joining the various particles to the centre of the earth ; but since the centre of the earth is about 4,000 miles away from points on its surface, we may consider that all the lines drawn to the earth's centre from various points of a body of finite size on the earth's surface are **parallel** lines ; and we can therefore find the weight of any body or system of particles in the same way as we find the resultant of a system of parallel forces. The point at which this resultant acts—that is, the centre of these parallel forces—is called the **centre of gravity** of the body or system of particles.

Hence the weights of all the particles of which a body is composed may be replaced by a single weight, equal to their sum, and acting through the centre of gravity of the body.

We therefore have the following definition :

The Centre of Gravity of a Body is the point at which the whole weight of the body may be supposed to be collected.

When we consider the mass of the body only, this point is called the **centre of mass**.

Since the centre of gravity is the point at which the resultant weight of a body may be supposed to act, it follows that the line of action of the resultant weight of a body must pass through its centre of gravity in whatever position the body is placed.

Now a number of lines cannot meet together in more than

one point ; hence no body can have more than one centre of gravity. If this point be supported, the body or system will rest in any position.

82. If a heavy body be suspended by a single string, or by any number of strings, to a fixed point, the centre of gravity must be vertically beneath this point.

For all the forces which act on the body may be compounded into two—the resultant of all the tensions which must pass through the fixed point, and the resultant weight of the body acting vertically downwards through the centre of gravity. If there is equilibrium these two must be equal and opposite, therefore the point of suspension and the centre of gravity must be in the same vertical line.

This gives us a useful practical method of determining the centre of gravity of a body. Suspend the body by a string, produce the direction of the string through the body ; this must also be the direction of the weight, therefore it must pass through the centre of gravity. Now suspend it from any other point and do the same thing. The point where these two lines meet must be the centre of gravity.

83. *If a body be placed upon a horizontal surface it will stand or fall, according as the vertical through its centre of gravity falls within or without its base.*

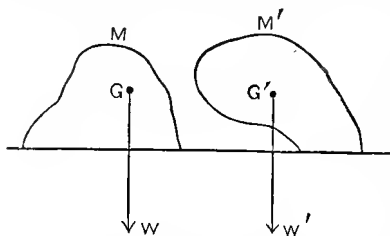


FIG. 39.

All the forces which act on the body may be compounded into two—the resultant weight acting vertically downwards through its centre of gravity, and the resultant pressure of the surface on the body.

Of these two the latter must act within the base of the body, and, if there is equilibrium, it must be equal and opposite to the former ; that is, the vertical through the centre of gravity must fall within the base.

If this vertical line fall outside the base, the body must

fall over towards the point where this vertical line cuts the surface.

If this line fall on the boundary of the base, the body will just stand, but will be on the point of falling over.

84. **Equilibrium** may be of three kinds—

Stable, Unstable, or Neutral.

Suppose a body to be in such a position that the forces acting upon it are in equilibrium, and that the body is then slightly displaced :

(1) If the forces now acting tend to make the body return to its original position, it is said to be in **stable** equilibrium.

(2) If these forces tend to make the body recede from its original position, it is said to be in **unstable** equilibrium.

(3) And if the body after displacement tends neither to return nor to recede, it is said to be in **neutral** equilibrium.

Examples :—

(1) A body supported at a point above its centre of gravity, *e.g.* a pendulum.

Bodies resting on a surface such that the vertical line through their centres of gravity falls well within the base.

(2) All bodies resting on very small bases, so that when displaced the weight acts in a line falling outside the base ; *e.g.* an egg balanced on its end, and all bodies which are *top-heavy*.

(3) All bodies which are supported at their centre of gravity, and which, therefore, rest in any position ; or bodies which, when displaced, still have their centre of gravity vertically above the point of support, *e.g.* a sphere on a horizontal table.

We will now proceed to find the centre of gravity of some bodies of simple shape and of uniform or homogeneous material.

Note.—Matter is said to be uniform or homogeneous when it is of the same density throughout—that is, when equal volumes of it are of equal mass and therefore of equal weight.

85. To find the **centre of gravity** of a **uniform material straight line** or of a **uniform rod**.

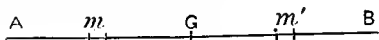


FIG. 40.

Let AB be the rod, G its middle point. The rod may be supposed to consist of a number of particles of equal weight. The resultant of the weights of any two equal particles m and m' equidistant from G must act through G . Hence the resultant weight of the rod must act through G , which must therefore be the centre of gravity.

86. To find the **centre of gravity** of a **parallelogram**.

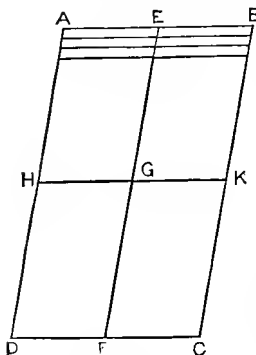


FIG. 41.

Let $ABCD$ be the parallelogram; E, K, F, H the middle points of the sides.

The parallelogram may be divided up into strips of material parallel to AB and DC .

The centre of gravity of each of these strips is at its middle point; but EF bisects all such strips; therefore the centre of gravity of each strip lies in the line EF ; hence the centre of gravity of the parallelogram lies in EF .

Similarly, by dividing the parallelogram into strips parallel to AD , the centre of gravity can be proved to lie in the line HK ; therefore it must lie in G , the point where EF and HK intersect. It can easily be proved by geometry that this point is also the point in which the diagonals intersect.

Similarly the centre of gravity of a circle must be its centre; for it can be divided into parallel strips, of which any pair equally distant from the centre are of equal length and of equal mass.

So, also, the centre of gravity of a sphere must be its centre.

87. To find the **centre of gravity** of a **uniform material triangle**.

Let ABC be the triangle, D the middle point of BC . The triangle may be divided into a number of thin material strips parallel to BC .

Let MN be one of these strips; then, since MN is parallel to BC , it can be proved that AD bisects MN ; therefore the centre of gravity of MN lies in AD .

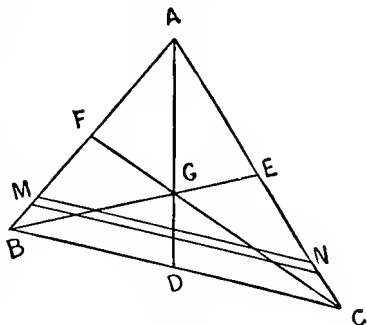


FIG. 42.

Hence the centre of gravity of every strip parallel to BC lies in AD ; therefore the centre of gravity of the triangle lies in AD .

Similarly, if E be the middle point of AC , it can be proved that the centre of gravity of the triangle must lie in BE .

Therefore the centre of gravity must lie at G , the point of intersection of AD and BE .¹

Note.—The straight lines joining the angular points of a triangle to the middle points of the opposite sides all pass through one point.

88. We have shown that the centre of gravity of a triangle lies at the point of intersection of certain lines in the figure; it will be often more convenient to know its exact position in one of these lines.

Join DE .

Then, since DE bisects CA and CB , therefore DE is parallel to AB and is equal to $\frac{1}{2} AB$.

Again, since ED is parallel to AB , the triangles GDE , GAB are similar;

$$\therefore \frac{GD}{DE} = \frac{GA}{AB}.$$

But $DE = \frac{1}{2} AB$;

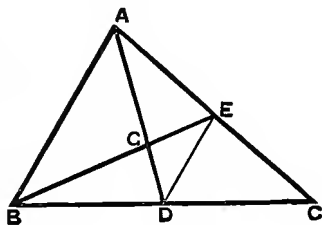


FIG. 43.

$$\therefore GD = \frac{1}{2} GA;$$

$$\therefore GD = \frac{1}{3} AD.$$

¹ A thin sheet or plate of material is often called a lamina.

Similarly, also, $GE = \frac{1}{3} BE$.

Hence, to find the centre of gravity of a triangular lamina, join any angular point A to the middle point of the opposite side D, and measure off a distance DG equal to $\frac{1}{3} AD$.

89. *The centre of gravity of a triangle coincides with the centre of gravity of three equal particles placed at its angular points.*

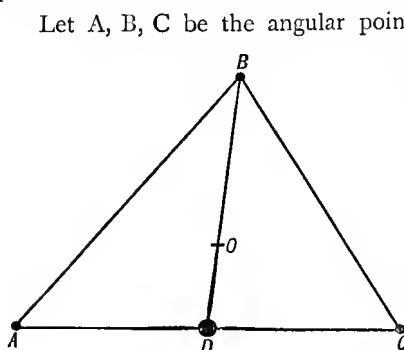


FIG. 44.

Let A, B, C be the angular points of a triangle, and let particles of equal weight w be placed there. Bisect AC at D. The resultant of the equal weights w at the points A and C is $2w$, acting at the point D.

The resultant of the weight w at B and $2w$ at D is $3w$ acting at a point O, such that

$$w \cdot OB = 2w \cdot OD;$$

$$\text{i.e. } OB = 2 OD.$$

Hence O coincides in position with the centre of gravity of the triangular lamina.

We can therefore suppose that the weight w of a uniform triangular table may be replaced by three weights, each equal to $\frac{1}{3} w$, at its angular points.

Hence we have the following result :

If a uniform horizontal triangular table rest on legs at its angular points, the pressure on each leg is the same.

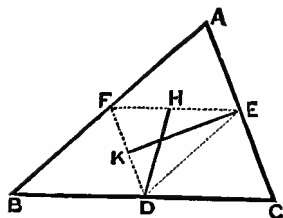


FIG. 45.

90.* *To find the centre of gravity of a uniform wire bent into the shape of a triangle.*

Let ABC be the triangular wire ; D, E, F the middle points of the sides, and let the lengths of

the sides be a, b, c . Then the lengths of EF, FD, DE are $\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c$ respectively.

Let w be the weight of a unit of length of the wire ; then the weight of $BC = wa$ acting at D , its middle point.

Hence the weight of the wire may be replaced by the three weights wa, wb, wc , acting at the points D, E, F respectively.

Now the centre of gravity of the weights wb at E and wc at F is at a point H in the line EF such that

$$wb \times EH = wc \times FH ;$$

$$\text{i.e. } \frac{EH}{FH} = \frac{w \cdot c}{w \cdot b} = \frac{c}{b} = \frac{ED}{FD} ;$$

therefore HD bisects the angle EDF (Euclid VI. 3), and the centre of gravity of the wire must lie in this line.

Similarly it can be shown that it must lie in the line EK which bisects the angle DEF , or in the bisector of the angle DFE .

Therefore these three lines all pass through one point, which is the centre of gravity of the wire. This point is the centre of the circle inscribed in the triangle DEF .

(Euclid IV. 4.)

91.* To find the centre of gravity of a pyramid on a triangular base.¹

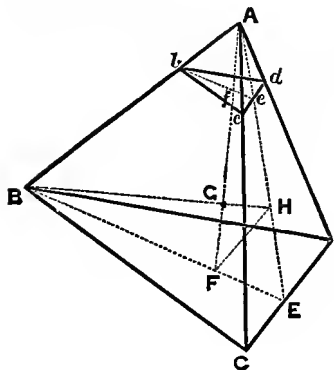


FIG. 46

Let $ABCD$ be the pyramid ; it may be considered as made up of a large number of triangular laminæ parallel to the base BCD .

Let bcd be one of these.

Bisect CD in E . Join AE , cutting cd in e ; e is the middle point of cd .

Join BE and cut off EF equal to $\frac{1}{3}BE$; F is the centre of gravity of the lamina BCD .

Join be . The plane ABE contains the line be .

Join AF , cutting be in f .

¹ Such a pyramid is usually called a 'tetrahedron.'

Then, by similar triangles,

$$\frac{ef}{EF} = \frac{Af}{AF} = \frac{fb}{FB}.$$

$$EF = \frac{1}{2} FB; \therefore ef = \frac{1}{2} fb;$$

$\therefore f$ is the centre of gravity of the lamina bcd ;

$\therefore$ the centre of gravity of the pyramid lies in the line AF .

Similarly, if H be the centre of gravity of the face ACD , it can be proved that the centre of gravity of the pyramid lies in the line BH . Now the plane ABE contains both the lines AF and BH . Therefore the centre of gravity of the pyramid lies at the point G , where AF and BH intersect.

Again, since $EH = \frac{1}{3} EA$, and $EF = \frac{1}{3} EB$,

$\therefore FH$ is parallel to BA and is $= \frac{1}{3} BA$;

$\therefore$ the triangles GFH , GAB are similar;

$$\therefore \frac{GF}{FH} = \frac{GA}{AB}.$$

$$\text{But } FH = \frac{1}{3} AB; \therefore GF = \frac{1}{3} GA; \\ \therefore FG = \frac{1}{4} AF.$$

Similarly, $HG = \frac{1}{4} HB$.

Hence, to find the centre of gravity of a pyramid, join the vertex A to F , the centre of gravity of the base, and measure a distance $FG = \frac{1}{4} FA$.

As in the case of the triangle, we can prove that the centre of gravity of a pyramid coincides with the centre of gravity of four particles of equal mass placed at the angular points.

92.* We can now find the centre of gravity of a pyramid on any base, polygonal or curvilinear, as follows :

Take any point within the base ; if the base is polygonal, we can divide the base into as many triangles as it has sides, by joining this point to the angular points of the base ; if the base is curvilinear, we can imagine the curve to

be made up of an infinite number of very short straight lines, and we can then divide the base up into a very large number of triangles with very small bases.

By joining the vertex of the pyramid to the point within the base, and also to the other angular points of all these triangles, we shall split up the pyramid into a number of pyramids, all having a common vertex and triangular bases in the same plane.

If we now draw a plane parallel to the base, and at a distance from the base equal to one-fourth of the distance of the vertex from the base, this plane will contain the centres of gravity of each of these pyramids on triangular bases.

Hence the centre of gravity of the whole pyramid must lie in this plane.

But by dividing the pyramid into thin laminæ all parallel to the base it can be shown, as before, that the centre of gravity of each lamina lies in the line joining the vertex to the centre of gravity of the base. Hence it must lie at the point where this line cuts the plane which contains the centre of gravity.

Therefore, to find the centre of gravity of a pyramid on any base, or of a cone on a circular base or on any curvilinear base, join the vertex to the centre of gravity of the base, and measure off from the base end a distance equal to one-fourth of the length of this line.

93. We will now apply the results of Article 62 to find the centre of gravity of a system of particles in a straight line or in the same plane.

(1) In a straight line.

Let w_1, w_2, w_3 , &c., be the weights of the particles; x_1, x_2, x_3 , &c., their distances from a fixed point O in the line.

$\bar{x}$ the distance of their centre of gravity from O.

Then,
$$\bar{x} = \frac{\sum (w \cdot x)}{\sum (w)}$$

(2) In the same plane.

Take any point O in the plane, and through O draw two lines x O x' , y O y' at right angles to each other.

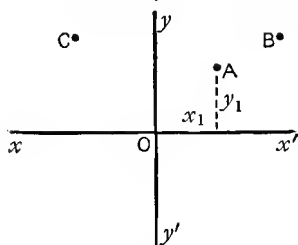


FIG. 47.

Then if $w_1, w_2, w_3, \&c.$, be the weights of the particles at the points A, B, C, &c., x_1, y_1 the co-ordinates of A, x_2, y_2 the co-ordinates of B, and so on, and $\bar{x}, \bar{y}$ the co-ordinates of the centre of gravity,

$$\bar{x} = \frac{\sum (w \cdot x)}{\sum (w)}, \quad \bar{y} = \frac{\sum (w \cdot y)}{\sum (w)}.$$

These results might also be derived from the principle that the moment of the resultant of any number of forces about a point or about a straight line is equal to the sum of the moments of its components.

Example 1.—Masses of 1 lb., 2 lbs., 3 lbs., 4 lbs., and 5 lbs. are placed in a straight line at distances 1 foot apart. Find their centre of mass.

Take the position of the 1 lb. mass as the fixed point. Then, with the usual notation,

$$\begin{aligned} \bar{x} &= \frac{1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4}{1 + 2 + 3 + 4 + 5} \\ &= \frac{40}{15} = \frac{8}{3} = 2\frac{2}{3}; \end{aligned}$$

i.e. the centre of gravity is 2 ft. 8 ins. from O.

Example 2.—To find the distance of the centre of gravity of a triangle from a point in its plane, the co-ordinates of the angular points with reference to rectangular axes through the fixed point being (1, 2), (2, 1), (2, 2).

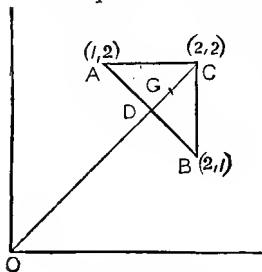


FIG. 48.

If W be the weight of the triangle, it can be replaced by three weights, each $= \frac{W}{3}$, at its angular points.

Then, with the usual notation,

$$\bar{x} = \frac{\frac{W}{3}(1) + \frac{W}{3}(2) + \frac{W}{3}(2)}{\frac{W}{3} + \frac{W}{3} + \frac{W}{3}} = \frac{\frac{5}{3}W}{W} = \frac{5}{3}.$$

So, also, $\bar{y} = \frac{5}{3};$

therefore, if r be the distance from O,

$$\begin{aligned} r^2 &= \left(\frac{5}{3}\right)^2 + \left(\frac{5}{3}\right)^2; \\ \therefore r &= \frac{5}{3} \sqrt{2} = \frac{5}{6} (2 \sqrt{2}) \\ &= \frac{5}{6} OC. \end{aligned}$$

Example 3.—Masses of 1 lb., 2 lbs., 3 lbs., 4 lbs. are placed at the angular points of a square. Find the distance of their centre of mass from the 1 lb.

Let ABCD be the square, with the masses 1, 2, 3, 4 at the points A, B, C, D respectively.

Let a be the side of the square.

Take A as origin, AB as axis of x , AD as axis of y .

Then the co-ordinates of A, B, C, D are $(0, 0)$, $(a, 0)$, (a, a) , $(0, a)$ respectively.

Then, if $\bar{x} \bar{y}$ be the coordinates of G, the centre of mass,

$$\bar{x} = \frac{1 \cdot 0 + 2 \cdot a + 3 \cdot a + 4 \cdot 0}{1 + 2 + 3 + 4} = \frac{5a}{10} = \frac{1}{2}a;$$

$$\bar{y} = \frac{1 \cdot 0 + 2 \cdot 0 + 3 \cdot a + 4 \cdot a}{1 + 2 + 3 + 4} = \frac{7a}{10} = \frac{7}{10}a.$$

If r be the distance of G from A

$$\begin{aligned} r^2 &= \bar{x}^2 + \bar{y}^2 = \frac{25}{100}a^2 + \frac{49}{100}a^2 = \frac{74}{100}a^2; \\ \therefore r &= \frac{\sqrt{74} \cdot a}{10}. \end{aligned}$$

94. If a body be made up of two parts whose centres of gravity are known, the centre of gravity of the whole can be found as follows :

Let G_1, G_2 be the centres of gravity of the two parts whose weights are w_1, w_2 respectively.

Then, if G be the centre of gravity of the whole body, it is the point at which the resultant of w_1 and w_2 acts; therefore, by Art. 57, it must lie in the line joining $G_1 G_2$ in such a position that

$$w_1 \times G G_1 = w_2 \times G G_2,$$

from which equation $G G_1$ or $G G_2$ can be found.

If a body whose weight and centre of gravity are known has a known portion removed from it, to find the centre of gravity of the remainder.

Let W be the weight of the body, G its centre of gravity.

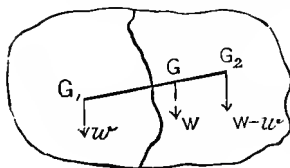


FIG. 49.

Let w be the weight and G_1 the centre of gravity of the part which is taken away; then G_2 , the centre of gravity of the remainder, must lie in $G_1 G$ produced, at such a distance from G that

$$(W - w) \times G G_2 = w \times G G_1;$$

$$\therefore G G_2 = \frac{w}{W - w} \times G G_1.$$

Example 1.—A T-shaped figure is made up of two rectangular pieces of cardboard, 5×2 sq. ins. and 4×3 sq. ins., as shown in the figure. Find its centre of gravity.

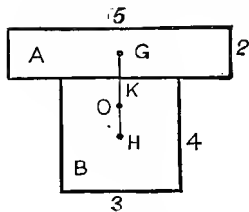


FIG. 50.

Call the pieces A and B.

Let G and H be their centres of gravity respectively.

Then GH bisects their coincident sides in K .

Let O be the centre of gravity of the figure.

Then weight of $A \times GO$ = weight of $B \times HO$.

Now weight of A : weight of B :: area of A : area of B ;

$$\therefore 10 : 12 ::$$

$$\therefore 10 GO = 12 HO;$$

$$\therefore 5 GO = 6 HO;$$

$$\therefore 5 GO = 6 (HG - GO);$$

$$\therefore 11 GO = 6 HG.$$

Now $GK = 1$ inch and $KH = 2$ inches ; $\therefore HG = 3$ inches ;

$$\therefore 11 GO = 18 \text{ ins. ;}$$

$$\therefore GO = 1\frac{7}{11} \text{ in. ;}$$

$$\therefore KO = \frac{7}{11} \text{ in.}$$

Example 2.—To find the centre of gravity of a quadrilateral of which two sides are parallel and their lengths in a given ratio.

Let $\frac{AB}{CD} = r$. Join BD . Then

the areas of triangles of the same altitude are proportional to their bases ;

$$\therefore \frac{\Delta ABD}{\Delta BDC} = \frac{AB}{DC} = r.$$

Let $3w =$ weight of the triangle BDC .

Then $3rw =$ weight of the triangle ABD .

But the weight of the triangle can be replaced by three equal weights at its angular points.

Hence, BDC can be replaced by w at the points B, D, C ,
and ABD „ „ rw „ A, B, D .

Therefore the weight of the quadrilateral can be replaced by the weights,

rw at A , $w + rw$ at B and at D , and w at C .

Hence the position of the centre of gravity can be found by means of Article 93.

Example 3.—A lamina is made up of a square whose side is 2 inches and a triangle, on the same base but on opposite sides of it ; find the height of the triangle in order that the figure should balance when this common base is supported.

Let $ABCD$ be the square, ABE the triangle whose altitude EN is required. Call it h inches.

Let G, H be the centres of gravity of the square and the triangle respectively.

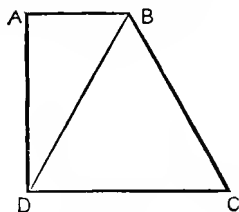


FIG. 51.

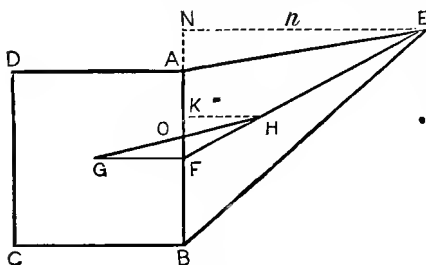


FIG. 52.

The centre of gravity of the figure must lie in GH , and by the question it must lie in AB ; therefore it must lie at the point O where GH cuts AB ;

$\therefore$ weight of triangle $\times HO$ = weight of square $\times GO$.

Now the weights of the triangle and the square are proportional to their areas:

the area of the square = $(AB)^2 = 4$ sq. ins.,
and the area of the triangle = $\frac{1}{2} AB \cdot EN = \frac{1}{2} \cdot 2 \cdot h = h$ sq. ins.;

$$\therefore h \times HO = 4 \times GO.$$

Now
$$\frac{HO}{GO} = \frac{HK}{GF};$$

$$\therefore h \times HK = 4 \times GF \quad \dots \dots (1)$$

Now
$$\frac{HK}{EN} = \frac{HF}{EF} = \frac{1}{3};$$

$$\therefore HK = \frac{1}{3} EN = \frac{1}{3} h, \text{ and } GF = 1 \text{ in.};$$

$$\therefore h \times \frac{h}{3} = 4$$

$$h^2 = 3 \times 4$$

$$\therefore h = 2\sqrt{3} \text{ ins}$$

$$= \sqrt{3} \text{ (the side of the square).}$$

Note.—The equation (1) might have been obtained at once from the consideration that, since the whole figure balanced about AB , the moments of the weights of the two parts about AB must be equal in magnitude.

Hence, weight of triangle $\times HK$ = weight of square $\times GF$.

Example 4.—A square $ABCD$, whose centre is O , has the portion AOB cut away. Find the centre of gravity of the remainder.

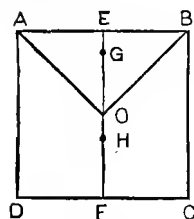


FIG. 53.

Through O draw EF bisecting the sides AB, CD .

Then the centre of gravity of AOB lies at G , such that $OG = \frac{2}{3} OE$, and the centre of gravity of the square lies at O ;

$\therefore$ the centre of gravity of remainder lies at H in GO produced, such that

$$\begin{aligned} \text{weight of part } AOB \times OG \\ = \text{weight of remainder} \times OH. \end{aligned}$$

Now the triangle $A O B = \frac{1}{4}$ (the square);

$\therefore$ the remainder = 3 (the triangle $A O B$)

$$\therefore O G = 3 O H ;$$

$$\begin{aligned}\therefore O H &= \frac{1}{3} O G \\ &= \frac{1}{3} \left(\frac{2}{3} O E \right) = \frac{2}{9} O E \\ &= \frac{1}{9} \text{ (side of the square).}\end{aligned}$$

Examples.

1. Weights of 1 lb. and 5 lbs. are placed 8 ins. apart ; where is their centre of gravity ?

2. Weights of W and $8 W$ are placed 5 ins. apart ; where is their centre of gravity ?

3. Masses of 1, 1, 1 and 2 lbs. are placed at the angular points of a square ; find their centre of mass.

4. Weights of 2, 2, 2 and 1 lb. are placed at the angular points of a square ; find their centre of gravity.

5. Five equal weights are placed in a straight line at distances apart of 1, 2, 3, 4 ins. respectively. Show that the centre of gravity lies at a point whose distances from the outside weights are in the ratio 2 : 3.

6. Masses of 3 lbs., 5 lbs., 7 lbs., 9 lbs. are in a straight line at intervals of a foot ; find their centre of mass.

7. If there are a number of heavy particles situated at fixed points, and another heavy particle move along a given straight line, prove that the centre of gravity of the whole system will also move along a certain straight line.

8. A triangular lamina is suspended from one of its angular points, and rests with the opposite side horizontal ; show that the triangle is isosceles.

9. O is the centre of gravity of a triangle $A B C$; forces are represented by $O A$, $O B$, $O C$; show that they are in equilibrium.

10. The sides $A B$, $A C$ of a triangle $A B C$, right-angled at A , are respectively 18 and 12 inches long. Find the distance of the centre of gravity from C .

11. If the base of a triangle be fixed, and its vertex always lie in a straight line, prove that the centre of gravity will also lie in a certain straight line.

12. D, E, F are the middle points of the sides of a triangle ABC; find the centre of gravity of the triangle DEF.

13. ABCD is a square plate, E and F being the middle points of the sides AB and BC; the plate is bent along EF, so that the triangle EBF lies flat on the other part of the plate. Find the centre of gravity.

14. A circular board has two circular holes cut in it, the centres of these holes being at the middle points of two radii of the board at right angles to each other. Find the centre of gravity of the remainder if each of the pieces cut out be equal to one-ninth of the board.

15. A circular board of radius a has a hole of radius b cut out of it. Show that the centre of gravity of the remainder must lie within a circle whose radius is $\frac{b^2}{a+b}$.

16. Two equal rods are fixed on a circular board so as to coincide with the chords of two adjacent quadrants. If the weight of each rod be equal to the weight of the board, find the centre of gravity of the whole.

17. A triangular table of 3 lbs. weight, whose sides are 3, 4, and 5 ins., is supported by legs of equal length at its angular points. Find the pressure on each leg.

18. If the centre of gravity of three equal particles placed on the circumference of a circle is at the centre of the circle, prove that the particles are equally distant from each other.

19. ABC is a uniform triangular plate of weight $3w$, weights of $5w$, w and w are placed at A, B, C respectively. Find the point about which the triangle will balance.

20. An isosceles triangle is suspended (1) from the vertex, (2) from one of the equal angles. The angle between the two positions of the base is 60° . Find the angles of the triangle.

21. A right-angled isosceles triangle is suspended (1) from the right angle, (2) from one of the equal angles. Find the tangent of the angle between two positions of any one of its sides.

22. A lamina in the shape of a right-angled triangle, such that one of the sides containing the right angle is three times the other, is suspended freely from the right angle; prove that in equilibrium the hypotenuse makes an angle $\sin^{-1} \frac{3}{5}$ with the vertical.

23. A rectangle is divided into four parts by lines joining the middle points of opposite sides ; one of them is cut away ; find the centre of gravity of the remainder.

24. A B C D is a square, O its centre, E, F the middle points of A B, A D ; A E F is cut away ; find G, the centre of gravity of the remainder.

25. An equilateral triangle and a square are described on opposite sides of the same base. Find the centre of gravity of the figure so formed.

26. Two squares, of which one is four times the other, are placed so that the sides about an angular point of the one are co-linear with those about an angular point of the other. Find the centre of gravity of the figure so formed.

27. A quarter of a square is cut away by lines drawn from the centre to two adjacent corners. Find the centre of gravity of the remainder.

28. In a quadrilateral A B C D the sides A B, A D are 15 ins., and B C . C D are 20 ins. If B D = 24 ins., find the distance of the centre of gravity of the quadrilateral from A.

29. A figure is made up of a square and an isosceles triangle on opposite sides of the same base. Find the relation between the altitude and base of the triangle in order that the centre of gravity of the figure should lie in the common base.

30. Masses of 1 lb., 2 lbs., and 3 lbs. are placed at the angular points A, B, C of a triangle. Find their centre of gravity, and show that it lies on the line joining the middle points of C A, C B.

31. Masses of 2, 6, 4, 5, 3, 7 lbs. are placed at the angular points of a hexagon taken in order. Show that their centre of gravity is at the centre of the hexagon.

32. Particles of 2 lbs., 1 lb., 2 lbs., 3 lbs. are placed at A, B, C, D respectively, the angular points of a square. Find the distance of the centre of gravity from the centre O.

33. Squares are described on the three sides of an isosceles right-angled triangle, outside the triangle. Find the centre of gravity of the figure so formed.

34. A B C is a triangle and A' a point inside it. Find the position of A' when it is the centre of gravity of the area which lies between the two triangles A B C, A' B C.

35. Where must a circular hole of 1 ft. radius be punched out of a circular disc of 3 ft. radius, so that the centre of gravity of the remainder may be 2 ins. from the centre of the disc?

36. How would you place a brick, whose length, breadth, and thickness were all different, on a rough inclined plane so that it would be least likely to tumble over?

37. A brick whose dimensions are $8 \times 4 \times 3$ ins. rests on a rough plane in such a way that it cannot slip, and the plane is tilted about a line parallel to one edge of the brick. Find the greatest and least angles of inclination for which the brick will just not upset.

38. Masses of 3 lbs., 5 lbs., 7 lbs., 9 lbs. are at the angular points of a square $A B C D$. Find their centre of mass.

39. A square lamina $A B C D$ is suspended from A ; weights of 4 lbs., 5 lbs., 16 lbs. are suspended from B , C , D respectively, and it hangs so that A is vertically above the middle point of $C D$. Find the weight of the square.

40. A uniform rod $A B$, 8 lbs. weight and 20 ins. long, is suspended horizontally by two vertical strings attached to the points B and C , C being distant 4 ins. from A . Find the greatest weight which can be attached to the end A without disturbing the equilibrium of the rod.

41. G is the centre of gravity of a triangle $A B C$; a line is drawn through G parallel to $B C$ cutting $A B$, $A C$ in P and Q . Show that the centre of gravity of $P B C Q$ divides $G D$ in the ratio of 8 : 7, D being the middle point of $B C$.

42. The sides of a quadrilateral lamina taken in order are 3, 5, 4, 10 respectively; the side 5 is parallel to 10. Find the distance of the centre of gravity of the quadrilateral from each of the sides 3 and 4.

43. A heavy body is suspended by a number of strings, all attached to a fixed point. Show that the centre of gravity must be vertically beneath this point.

44. A flat triangular board $A B C$, right-angled at A , stands with its plane vertical and its side $A C$ on a horizontal plane; D is the middle point of $A C$. If the portion $B A D$ be cut away, will the remainder stand or fall?

45. The wheels of a hay-cart are 10 feet apart, and the centre of gravity of the cart and load is 12 feet above the ground and mid-

way between the wheels. How much could either wheel be raised without the cart falling over ?

46. $ABCD$ is a square standing on CD as base in a vertical plane, E is a point in CD , and the triangle ACE is cut away. Find the least length of DE in order that the remainder should not fall.

47. The radius of the base of a cone is to the altitude as $2 : 15$, the cone is placed on its base on a smooth inclined plane, and is kept from slipping by a string fastened to a point in the plane and to the rim of the base. Find the greatest ratio of the height of the plane to its base which is consistent with equilibrium.

48. A brick is laid with a quarter of its length projecting over the edge of a wall ; a brick and a quarter brick are laid on the first with a quarter of its length over the edge of the first brick ; a brick and a half laid on this, and so on. Prove that four such courses can be laid in the above manner, but that if a fifth course be added the mass will topple over.

49. $ABCD$ is a rectangle, E the middle point of CD ; the triangle ADE is cut away. Find the centre of gravity of the remainder.

50. The lengths of the sides of an isosceles triangle are $2a$, $2a$, $2b$; on each side an equilateral triangle is described. Find the distance of the centre of gravity of the whole figure from the side $2b$.

Verify your result by applying it to the case of an equilateral triangle.

CHAPTER VII

GENERAL CONDITIONS OF EQUILIBRIUM

95. *Two* forces cannot be in equilibrium unless they are equal and opposite in the same straight line. Otherwise, if they act in the same plane, they have a single resultant or form a couple.

If they do not act in the same plane they cannot be in equilibrium, nor can they be replaced by a single force or a couple.

Consequently, *if three forces acting on a body are in equilibrium, they must all act in the same plane*, for otherwise it would be impossible for the third force to balance the resultant of the other two.

96. If three forces acting on a body in one plane are in equilibrium, they must all be parallel or they must all pass through one point.

For consider any two of them—their lines of action must be parallel or they must meet in a point.

If they are parallel, they must have a resultant which is parallel to them; and, since there is equilibrium, the third force must be equal and opposite to this resultant—that is, it must be parallel to them.

If the two are not parallel, produce their lines of action to meet at a point; then they have a resultant which passes through this point; and, since there is equilibrium, the third force must be equal and opposite to this resultant; that is, it must pass through the point.

Note.—If there is equilibrium, no two of the three forces can form a couple.

97. *To find the conditions of equilibrium of any number of forces which act on a body at one point.*

The algebraical sum of the resolved parts of the forces in any two directions through the point must each be zero. (Art. 55.)

It is not necessary that these directions should be at right angles to each other, but it is generally more convenient to take them so.

This condition is sufficient for such a system of forces. For since all the forces pass through one point they cannot reduce to a couple ; hence, if the resultant vanishes, the forces must be in equilibrium.

98. *To find the conditions of equilibrium of any number of co-planar forces which act on a rigid body.*

(1) If the forces are parallel.

The algebraical sum of the forces must be zero, *and* the algebraical sum of the moments of the forces about any point in the plane must also be zero. (Art. 78.)

(2) If the forces are not parallel.

The algebraical sum of the resolved parts of the forces in two directions at right angles to each other must each be zero, *and* the algebraical sum of the moments of the forces about any point in the plane must also be zero. (Art. 79.)

99. The results of the previous articles might also have been obtained from Article 63.

We then proved that, if there is not equilibrium, the system reduces to a single force or a couple.

Now in part (2) of the last article, if the first condition holds, the resultant vanishes ; that is, the system cannot reduce to a single force.

If the second condition holds, there is no rotation ; that is, the system cannot reduce to a couple. Therefore, if both conditions hold, the system must be in equilibrium.

100. *A system of co-planar forces will be in equilibrium if the algebraical sum of the moments of the forces about any three points not in the same straight line vanishes in each case.*

Suppose A, B, C to be the three points.

We have seen that if the sum of the moments of any number of co-planar forces about any point in their plane vanishes, the forces must be in equilibrium, or the point must be in the line of action of their resultant. (Art. 71.)

Hence, in the above case, if the forces are not in equilibrium, A must be a point in the direction of their resultant.

Similarly, B and C must be points in the direction of the resultant.

This is impossible unless A, B, C are in the same straight line. Hence, if they are not in the same straight line, the system of forces must be in equilibrium.

101. If one point of a rigid body be fixed, the forces will be in equilibrium if the algebraical sum of the moments of the forces about the fixed point vanishes.

This is often found to be an extremely useful condition, since in many cases the reaction of the fixed point is unknown, and if we consider the rotation about that point we shall get a relation between the other forces excluding this unknown reaction.

In solving problems it will be found that when three forces acting on a body are in equilibrium the position of equilibrium will be determined by Art. 96.

When more than three forces act on a body, and are in equilibrium, we apply the conditions laid down in Art. 98; of these sometimes one set of conditions is sufficient to solve the problem without the other.

The latter condition is usually the most useful of the two, provided that sufficient care is taken in choosing the points about which we take moments.

For example, if four forces are in equilibrium and we take moments about the point where two of them intersect, we shall get a simple relation between the other two.

Note.—The expression *take moments about a point* is a phrase meaning, *express the fact that the algebraical sum of the moments of the forces about that point is zero.*

102. In certain cases of equilibrium we shall often find it

convenient to consider the rotation of the body about a certain fixed line in it.

For example, if two points of a body be fixed, the body can only have one motion—namely, a rotation about the line joining these two points.

This line is then called the axis of rotation.

We shall only consider cases in which the forces which act upon the body are perpendicular to this fixed axis; in which cases the turning powers of the forces about the axis are measured by the product of the force and the perpendicular distance between its line of action and the fixed axis.

It is easy to see that this moment represents the total effect of a force on a body which is fixed in this manner. For let AB be the fixed axis, OC the perpendicular distance between AB and the line of action of P ; at C apply two equal and opposite forces P, P parallel to the direction of P . Then the original force P acting at O may be replaced by a single force P acting at C and a couple whose moment is $P \cdot OC$; of these the single force has no effect on the body since it acts through the fixed axis, and we have left the couple only. Hence, the only effect of P is a rotation which is measured by the product $P \cdot OC$.

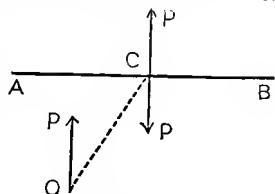


FIG. 54.

103. Before proceeding to work examples of forces in equilibrium, it will be useful here to consider in what manner various forces act.

In Article 6 we mentioned that the forces which affect a body's state of equilibrium must arise from some external cause or agency which produces a force of the nature of a push or a pull or an attraction. In all cases a force is one part of a mutual or reciprocal action, to which we give the name of *stress*. A push or pressure may be due (1) to the action of a rod on the body, (2) to the action of another body in contact with it, or (3) to the action of a hinge about which the body can turn.

In all cases of smooth bodies this mutual reaction must be

along the common normal to the surfaces ; for if there were any tangential action there could not be equilibrium, since there is no tangential force to counteract it.

In the case of a rod pushing a body, the pressure must act along the normal to the surface at the point of contact.

In the case of two surfaces in contact, the mutual pressure will be along the common normal at the point of contact of the two surfaces.



FIG. 55.

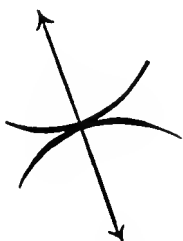


FIG. 56.

The direction of the reaction of a hinge will often be apparent from the conditions of the question ; but if it is not we must assume the magnitude and direction of this unknown reaction, or assume the magnitude of its components in two given directions at right angles, and find the value of these unknown components from the conditions given in the problem.

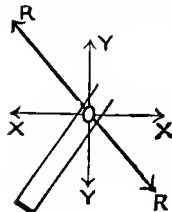


FIG. 57.

Then, with the usual notation, $R^2 = X^2 + Y^2$,

$$\text{and } \tan \theta = \frac{Y}{X}.$$

A tension may be due to the action of a string or rod on the body.

In this case every particle ab of the string or rod is kept in equilibrium by two equal and opposite forces t, t , which are in

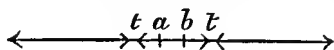


FIG. 58.

equilibrium, and are the halves of stresses whose other halves act on the next particle, and so on.

Hence the tension of the string is the same throughout its length.

Further, the tension of a string is not changed by passing the string over any smooth surface, such as a peg or pulley. For if we consider any small particle of the string, it is kept in equilibrium by the tensions at its ends and the reaction of the surface along the normal.

Since the two tensions at the extremities of the particle make equal angles with the normal, we see, by resolving them along and $\perp$ to this normal, that they must be equal.

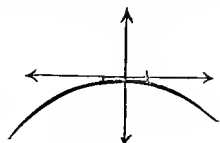


FIG. 59.

The force of attraction between two bodies always acts along the line joining their centres of mass. The particular case of gravity we have fully considered in the previous chapter.

Note.—The student has several times been warned against confounding mass with weight, and he has been told that 1 lb. is a mass and not a weight; yet for the sake of brevity we do use the expressions weight of 6 lbs., or 6 lb. weight; but we always mean by such expressions, a weight equal to the weight of the mass 6 lbs., or the weight of the mass 6 lbs.

Example 1.—A uniform rod whose weight is W is movable about a hinge at one end; it is kept in equilibrium in a position making an angle of 30° with the horizontal by a force making an angle of 30° with the rod at its other end.

Determine the reaction of the hinge and the direction of its line of action.

Let AB be the rod, hinged at B ; its weight W acts vertically downwards through the middle point C .

Let F be the force at A , R the reaction of the hinge.

Let F and W meet at O ; then, since there is equilibrium between the forces F , W , and R , R must also pass through O .

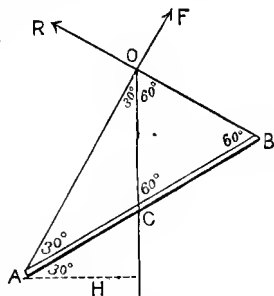


FIG. 60.

It is obvious from the geometry of the figure that since

$\angle OAC = 30^\circ$ and $\angle CAH = 30^\circ \therefore$ also $\angle COA = 30^\circ$ and $\angle OCB = 60^\circ$.

Also $CO = CA = CB$; $\therefore \angle COB = \angle CBO = 60^\circ$;

$\therefore R$ makes an angle of 60° with the rod.

To find R , we have by Lami's theorem, Art. 51,

$$\frac{R}{\sin 150^\circ} = \frac{F}{\sin 120^\circ} = \frac{W}{\sin 90^\circ};$$

$$\therefore \frac{R}{\frac{1}{2}} = \frac{W}{1};$$

$$\therefore R = \frac{1}{2} W.$$

This result could also be obtained by taking moments about A .

Example 2.—A ladder is placed with one end on a smooth horizontal plane and its other end against a smooth vertical wall; show that it must slip down. If the ladder be uniform and of weight W , and is kept at rest at an angle θ with the wall by a man putting his foot against the lowest rung so as to apply a horizontal pressure at that point, find this pressure.

Let AB be the ladder. It is acted on by three forces—its weight W , and the pressures of the ground and the wall $\perp$ to the ground and wall respectively.

Call these pressures P and R .

Now P , R , and W cannot be in equilibrium because P and W are parallel and R is $\perp$ to them; therefore the ladder must slip down.

Let F be the horizontal force applied by the man.

Resolve the forces horizontally and vertically, then we have

$$P = W;$$

$$F = R.$$

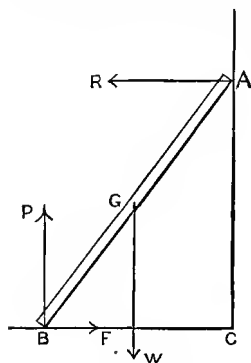


FIG. 61.

Take moments about B , and let $2l$ = length of the ladder:

$$R \cdot 2l \cos \theta = W \cdot l \sin \theta;$$

$$\therefore R = \frac{W \cdot \tan \theta}{2}; \quad \therefore F = \frac{W \tan \theta}{2}.$$

Note 1.—If R and W meet in O the resultant of P and F must pass through O .

Note 2.—The four forces P, W, R and F form two couples which are equal in magnitude and opposite in direction.

Example 3.—A sphere of weight W rests in contact with two inclined planes whose inclinations to the horizon are α, β . Find the pressures on the planes.

Let R, R' be the pressures at A and B the points of contact of the sphere with the planes.

Then R, R' must be $\perp$ to the planes, and make angles α, β with the vertical, and the three forces R, R' and W all pass through the centre of the sphere.

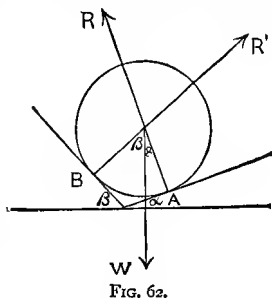


FIG. 62.

$\therefore$ by Lami's theorem,

$$R : R' : W :: \sin \beta : \sin \alpha : \sin (\alpha + \beta) ;$$

$$\therefore R = \frac{\sin \beta}{W \sin (\alpha + \beta)} \text{ and } R' = \frac{\sin \alpha}{W \sin (\alpha + \beta)}.$$

Example 4.—A uniform beam of weight W is placed with one end on a smooth horizontal plane, and the other end rests on another plane inclined at an angle α to the horizon; to this end a string is fastened which passes over a pulley at the top of the plane and supports a weight w hanging freely. Show that the beam will rest in any position if $2w = W \sin \alpha$.

Let AB be the beam, R, R' the pressures of the planes on the ends A and B respectively.

Resolve vertically and horizontally

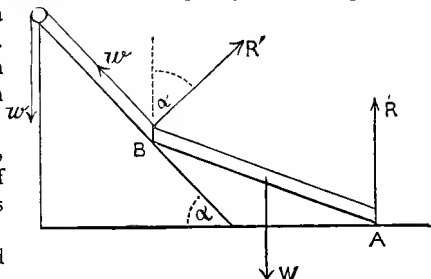


FIG. 63.

$$W = R + R' \cos \alpha + w \sin \alpha \quad \dots \quad (1)$$

$$R' \sin \alpha = w \cos \alpha \quad \dots \quad (2)$$

Take moments about B :

$$W = 2R \quad \dots \quad (3)$$

Eliminate R and R' by substituting their values from equations (2) and (3) in equation (1):

$$W = \frac{W}{2} + w \cdot \frac{\cos a}{\sin a} \cdot \cos a + w \sin a :$$

$$\frac{W}{2} = w \left(\frac{\cos^2 a}{\sin a} + \sin a \right) = \frac{w}{\sin a} ;$$

$$\therefore 2w = W \sin a.$$

Example 5.—A uniform rod AB , 18 feet long and of 20 lbs. weight, is hinged at A , and has a weight of 5 lbs. suspended from B . It is kept at rest by a string 13 ft. long, one end of which is attached to a point D on the rod 13 ft. from A , and the other end to a point O 10 ft. vertically above A . Find the magnitude and direction of the reaction of the hinge.

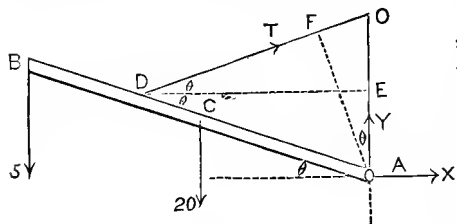


FIG. 64.

Let θ be the angle the rod makes with the horizontal.

Draw $DE \perp$ to OA .

Then, since $DO = DA$, DE bisects the angle DOA ;

$$\therefore ODE = EDA = \theta.$$

If AF is $\perp$ to OD , $OAF = \theta$.

Let X and Y be the horizontal and vertical components of the reaction of the hinge at A , and T the tension of the string.

Resolve vertically and horizontally

$$T \sin \theta + Y = 20 + 5 \quad \dots \quad (1)$$

$$T \cos \theta + X = 0 \quad \dots \quad (2)$$

Take moments about A :

$$T \cdot AF = 20 \cdot AC \cos \theta + 5 \cdot AB \cos \theta.$$

But

$$AF = OA \cos \theta ;$$

$$\therefore T \cdot OA = 20 \cdot AC + 5 \cdot AB ;$$

$$\begin{aligned} \therefore 10 T &= 20 \times 9 + 5 \times 18 \\ &= 270 ; \end{aligned}$$

$$\therefore T = 27 \text{ lbs. wt.}$$

Now
$$\sin \theta = \frac{OE}{OD} = \frac{5}{13};$$

$$\therefore \cos \theta = \frac{12}{13}.$$

From (1)

$$\begin{aligned} Y &= 25 - T \sin \theta \\ &= 25 - 27 \cdot \frac{5}{13} = 14 \frac{8}{13} \text{ lbs. wt.} \end{aligned}$$

From (2)

$$\begin{aligned} X &= -T \cos \theta \\ &= -24 \frac{12}{13} \text{ lbs. wt.,} \end{aligned}$$

which shows that X acts to the left, and not to the right, as drawn;

$\therefore$ if R be the reaction of the hinge,

$$R^2 = X^2 + Y^2;$$

$$\therefore R = \frac{375.6}{13} = 28.9 \text{ lbs. wt.}$$

If ϕ be the angle R makes with the axis of x ,

$$\tan \phi = \frac{Y}{X} = -\frac{190}{324} = -.586.$$

We find from the tables that ϕ lies between 149° and 150° ; that is, it makes an angle of about 60° to the left of A O.

Example 6.*—A smooth uniform rod whose length is $2l$ and weight W is at rest in a smooth hemispherical bowl whose radius is r , which is fixed with its axis vertical, one end of the rod projecting over the edge of the bowl. Find the conditions of equilibrium and the pressures of the bowl on the rod.

Since there are only three forces acting on the rod, they must all act in one plane; hence the rod must rest in the vertical plane through the centre O of the bowl.

Let the figure represent the section of the bowl and rod by this plane. Let A B be the rod resting against the bowl at the points A and D.

Let R, R' be the pressures at A and B respectively.

Then, since the pressures of smooth surfaces must be normal to the surfaces, R is perpendicular to the tangent at A, and must therefore pass through the centre O; R' is perpendicular to the rod A B,

Let R, R' meet in E ; then, since EDA is a right angle, AE must be a diameter of the circle; therefore E lies on the circumference of the completed circle. Since R, R' and W are in equilibrium they must all pass through one point; therefore C , the centre of gravity of the rod, must be vertically below E .

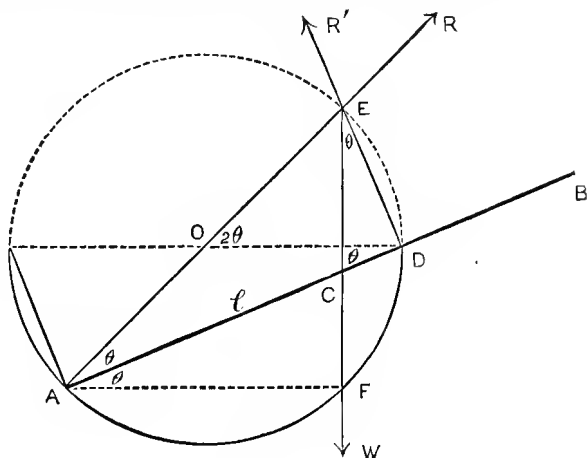


FIG. 65.

The above considerations give us the position of equilibrium.

Let θ = the angle of inclination of the rod to the horizon.

From the geometry of the figure we get the following :

since $ODA = \theta$, $OAD = \theta$, and $\therefore EOD = 2\theta$.

If AF be drawn perpendicular to the line of action of W , F must lie in the circle AED , since AFE is a right angle; and since AF is $\parallel$ to OD , $DAF = \theta$.

Resolve the forces along the rod and at right angles to the rod.

$$R \cos \theta = W \sin \theta \quad \dots \dots \dots (1)$$

$$R' + R \sin \theta = W \cos \theta \quad \dots \dots \dots (2)$$

Taking moments about A , we have

$$W \times AF = R' \times AD;$$

$$\therefore W \times l \cos \theta = R' \times 2r \cos \theta;$$

$$\therefore W \cdot l = 2 R' \cdot r \quad \dots \dots \dots (3)$$

From (1)

$$R = W \tan \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

substitute for R in (2);

$$\therefore R' = W \cos \theta - W \tan \theta \cdot \sin \theta \quad . \quad . \quad . \quad (5)$$

$$\therefore \frac{R'}{W} = \cos \theta - \frac{\sin^2 \theta}{\cos \theta} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta}.$$

But from (3)

$$\frac{R'}{W} = \frac{l}{2r};$$

$$\therefore \frac{l}{2r} = \frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta} = \frac{2 \cos^2 \theta - 1}{\cos \theta} \quad . \quad . \quad . \quad (6)$$

This is a quadratic for $\cos \theta$, from which we have

$$\cos \theta = \frac{l \pm \sqrt{32r^2 + l^2}}{8r}.$$

Since θ is less than 90°

$\therefore \cos \theta$ is positive ;

$\therefore$ the upper sign only of the root need be taken ;

$$\therefore \cos \theta = \frac{l + \sqrt{32r^2 + l^2}}{8r}.$$

This equation gives us the position of equilibrium of the rod when l and r are known.

The angle EOD cannot be greater than 90° ;

$\therefore \theta$ cannot be greater than 45° ;

$\therefore \theta$ must lie between 0° and 45° .

R and R' can be found from equations (4) and (5).

Case 1. If

$$\theta = 0,$$

$$R = W \tan \theta = 0,$$

and

$$R' = W \frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta} = W,$$

and

$$l = 2r.$$

Hence C and D coincide; that is, the rod balances on the point D .

Case 2. If

$$l = \frac{2r}{\sqrt{3}}, \cos \theta = \frac{\sqrt{3}}{2};$$

$$\therefore \theta = 30^\circ;$$

$$\therefore R = \frac{W}{\sqrt{3}} \text{ and } R' = \frac{W}{\sqrt{3}}.$$

Case 3. The least length the rod can have is when the extremity B just rests on the edge of the bowl.

In this case

$$\cos \theta = \frac{AD}{AE} = \frac{2l}{2r} = \frac{l}{r}.$$

Equating this with the value already obtained, we have

$$8l = l + \sqrt{32r^2 + l^2},$$

$$\therefore 3l^2 = 2r^2;$$

when
$$\frac{l}{r} = \frac{\sqrt{2}}{\sqrt{3}}.$$

In this position
$$\cos \theta = \sqrt{\frac{2}{3}}.$$

This gives the greatest value that θ can have, and in this case θ lies between 35° and 36° .

If the rod be any shorter B will fall within the bowl, and the rod will slip down into a horizontal position at the bottom of the bowl.

Hence, provided the length of the rod is less than $4r$ and greater than $\frac{2\sqrt{2}}{\sqrt{3}} \cdot r$, there is always one position, and one position only, in which the rod will rest, which position is given by the equation,

$$\cos \theta = \frac{l + \sqrt{32r^2 + l^2}}{8r}.$$

Note.—By Lami's theorem we get

$$R : R' : W :: \sin (180^\circ - \theta) : \sin (90^\circ + 2\theta) : \sin (90^\circ - \theta)$$

$$:: \sin \theta : \cos 2\theta : \cos \theta;$$

which relation includes equations (4) and (5). Equation (6) might have been obtained from the geometry of the figure instead of from equations (3) and (5).

Thus
$$\frac{AC}{AE} = \frac{l}{2r} = \frac{\sin AEC}{\sin ACE} = \frac{\cos 2\theta}{\cos \theta}.$$

Case 3 could be obtained by putting $l = r \cos \theta$, since AB cannot be less than AD, in which case equation (6) becomes

$$\frac{r \cos \theta}{2r} = \frac{2 \cos^2 \theta - 1}{\cos \theta}.$$

$$\therefore \cos^2 \theta = 4 \cos^2 \theta - 2$$

or

$$\cos^2 \theta = \frac{2}{3}.$$

Examples.

1. A uniform rod is suspended from a peg by two strings attached to each end, and of such a length that the angles between them and the rod are 30° and 60° . Find the tensions of the strings, the rod being 1 lb. weight.

2. A lamina in the shape of a rectangle, whose length is double its breadth, is suspended freely from a corner. Prove that in the position of equilibrium the diagonal makes with the vertical an angle $\sin^{-1} \frac{4}{5}$.

3. Two rods O A, O B of equal weight, whose lengths are a and b , are rigidly connected at O, so that the angle A O B is a right angle. The point O is attached to a string; if θ be the inclination of O A to the horizon, find $\sin \theta$.

4. In the preceding question if the rods be of equal length but their weights in the ratio $m : n$, show that $\tan \theta = \frac{m}{n}$.

5. A uniform wire is bent into two arms inclined at an angle of 120° , one arm being twice the length of the other. The wire is suspended from its angular point; find the position of equilibrium.

6. A uniform rod of length $2l$ is in equilibrium in a vertical plane. One end of the rod rests against a vertical wall, and the rod also rests on a prop whose distance from the wall is a . Find the angle the rod makes with the vertical.

7. A heavy uniform beam of weight W, resting on a smooth horizontal plane and against a smooth vertical wall, is held by a horizontal cord one-fourth the length of the beam attached to its middle point and to the wall. Find the tension of the cord and the pressures on the planes.

8. A ring of weight W slides freely on a string of length $a\sqrt{2}$ whose ends are fastened to two points at a distance a apart in a line making an angle of 45° with the horizon. Find the tension of the string in the position of equilibrium.

9. A uniform rod has a 5 lb. weight fixed at one end, and balances about a fulcrum 7 ins. from its middle point. If the 5 lb. weight be replaced by a 10 lb. weight the fulcrum must be moved 1 inch. Find the length of the rod and its weight.

10. A string is tied to two points. A ring weight W can slip freely along the string, and is pulled by a horizontal force P. If

the parts of the string when in equilibrium are inclined at 90° and 45° to the horizon, find the value of P .

11. A smooth horizontal bar is placed at a distance of 1 foot from a smooth vertical wall and parallel to it. A uniform rod is supported by the bar and the wall and is inclined at an angle of 30° to the vertical; show that its length is 16 feet.

12. A uniform rod 1 foot long and 3 lbs. weight has one end on a smooth table; a string is attached to a point 3 ins. from the other end, and is fastened to the ceiling so that the rod makes an angle with the table. Find the tension of the string.

13. One end of a uniform ladder 84 lbs. weight rests against a smooth vertical wall at a height 12 feet above the ground, and other end rests on the ground at a distance 10 feet from the wall. Find the pressure on the ground.

14. A uniform beam of mass 1 ton is suspended in a horizontal position by two ropes attached to its ends; one of the ropes, of the same length as the beam, is attached to a peg; the other rope passes over a pulley and is attached to a weight W ; the pulley is fixed in the same horizontal line as the peg, and at a distance from it equal to twice the length of the beam. Find W .

15. The two legs of a light step ladder are connected by a smooth joint at the top and a cord at the bottom. The ladder stands on a smooth floor with one leg, which is 3 feet long, vertical. A man of 11 st. weight stands on the other leg at a height of 2 feet above the ground. Find the pressure on the vertical leg. What is the tension of the cord?

16. A rod 5 feet long is inclined to the vertical at an angle of 45° , and stands on the ground resting against a smooth rail with 1 foot of its length projecting over the rail. Show by a figure the direction of the ground's action, and find the magnitude of this action if the mass of the rod be 6 lbs.

17. A smooth sphere of radius a and weight W is supported on a smooth plane at an angle of 30° to the horizon by a string, one end of which is fastened to a point in the plane and the other end to the surface of the sphere; in the position of equilibrium the string is horizontal. Find the length of the string and the pressure on the plane.

18. A heavy ladder is sustained at an inclination θ to the horizon by means of a force applied at the lowest round of the ladder

and by another force perpendicular to the ladder at a point half-way between the lowest round and the C.G. of the ladder. Find the direction of the first force, and compare the forces with the weight.

19. A heavy uniform beam PQ of length $2l$ has one end connected by means of a smooth ring with a vertical rod PAC , and rests on a vertical circle whose centre is C and whose radius $CA = a$. Find an equation for θ , the inclination of PQ to the horizon in the position of equilibrium.

CHAPTER VIII

MACHINES

04. A **Machine** is an instrument by means of which a force, which is applied at one point, is made available to do work or overcome resistance at some other point.

The simplest forms of machines are called the **Mechanical Powers**, and may be classified as follows :—

1. The **Lever**, which includes
 - (a) The **Steelyards**,
 - (b) The **Common Balance**,
 - (c) The **Wheel and Axle**.
2. The **Pulley** and **Systems of Pulleys**.
3. The **Inclined Plane**, which includes
 - (a) The **Screw**,
 - (b) The **Wedge**.

All machines consist of one or other of the above mechanical powers, or of combinations of them.

In the present chapter we shall consider only the simple machines, and find the relation between the forces which act upon them when they are at rest ; we shall also suppose that the various parts of the machines are perfectly smooth.

The primary object of a machine is to make a force overcome a resistance greater than itself.

The external force which we apply to the machine is called the **Power**, and is denoted by the letter **P** ; the resistance which this force can be made to balance is called the **Weight**, and is denoted by the letter **W**.

The ratio $\frac{W}{P}$ is called the **modulus** or **working power** of the machine, and is the measure of its **efficiency**.

If $\frac{W}{P}$ is $>$ unity—that is, if W is $>$ P —the machine is said to have **mechanical advantage**; but if $\frac{W}{P}$ is $<$ unity, the machine is said to have **mechanical disadvantage**.

It will be our object now to determine the efficiency of each of the simple machines when the power and the weight are in equilibrium.

The Lever.

105. The lever is a rigid rod capable of turning about a fixed point called the **fulcrum**. A lever may be *straight* or *bent*.

The power and the weight may act on opposite sides of the fulcrum or on the same side. Those portions of the lever between the fulcrum and the points of application of the power and the weight are called the *arms* of the lever.

Levers are divided into three classes, according to the relative positions of the fulcrum and the points of application of the power and the weight.

In **levers** of the **First Class** the power and the weight act on opposite sides of the fulcrum.

In the **Second Class** the power and the weight act on the same side of the fulcrum, but the power is further from the fulcrum than the weight.

In the **Third Class** the power and the weight are on the same side of the fulcrum, but the power is nearer to the fulcrum than the weight.

We will first consider the case of a straight lever without weight, in which the power and the weight are parallel and at right angles to the lever.

Let AB be the lever, F the fulcrum, P the power, W the weight, R the reaction of the fulcrum.



FIG. 66.

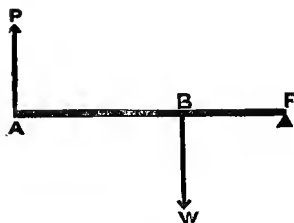


FIG. 67.

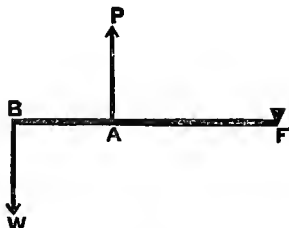


FIG. 68.

Fig. 66 represents the *first class* of levers, fig. 67 the *second class*, and fig. 68 the *third class*. In each case P and W tend to turn the lever in opposite directions round the fulcrum C , and the condition of equilibrium is obtained from the fact that the algebraical sum of the moments of P and W about F must vanish.

Hence

$$P \times FA = W \times FB ;$$

$$\therefore \text{efficiency} = \frac{W}{P} = \frac{FA}{FB}.$$

$$\text{In fig. 66} \quad R = P + W,$$

$$,, \quad 67 \quad R = W - P,$$

$$,, \quad 68 \quad R = P - W.$$

The lever has mechanical advantage or disadvantage, according as FA is $>$ or $< FB$.

In the *first class* there will be mechanical advantage or disadvantage according as the fulcrum F is nearer to the weight or to the power.

In the *second class* there will always be a mechanical advantage.

In the *third class* there will never be a mechanical advantage.

Examples of the first class of lever are the *common balance*, the *steelyard*, a *poker*, a *see-saw*.

A *pair of scissors* is a double lever of this class.

Examples of the second class are a *wheelbarrow*, an *oar* (the blade of the oar in the water being the fulcrum), a *tobacco-cutter*.

A *pair of nutcrackers* is a double lever of this class.

Examples of the third class, which is rarely made use of, are : a *catapult*, the *forearm of a man* with a weight in his hand (the elbow being the fulcrum).

A *pair of tongs* is a double lever of this class.

If the weight of the lever be w acting at the centre of gravity G , we must also take into account the moment $w \times FG$.

Thus, in fig. 66, if G is to the left of F ,

$$\begin{aligned} P \cdot FA &= W \cdot FB - w \cdot FG, \\ R &= P + W + w. \end{aligned}$$

In figs. 67 and 68

$$\begin{aligned} P \cdot FA &= W \cdot FB + w \cdot FG, \\ R &= (W + w) \sim P. \end{aligned}$$

106. If the lever be bent and the forces P and W are not parallel nor at right angles to the arms, the condition of equilibrium is still obtained by taking moments about the fulcrum.

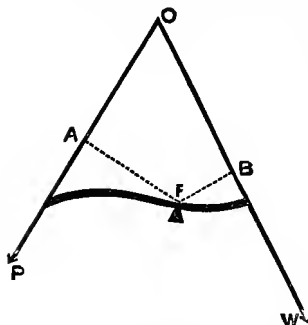


FIG. 69.

Draw FA , FB perpendicular to the directions of P and W respectively. Then

$$P \cdot FA = W \cdot FB.$$

In this case it is obvious that R , the reaction of the fulcrum, must pass through O , the point where the directions of P and W meet, since R , P , and W are in equilibrium.

107. If a heavy straight lever with weights attached to its extremities be in equilibrium in any one position, it will also be in equilibrium in any other.

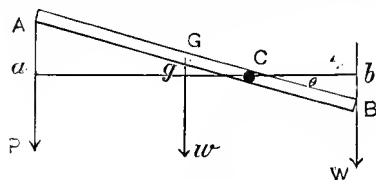


FIG. 70.

Let θ be the angle which AB makes with the horizontal.

Through C draw the common perpendicular to the directions of the

forces, cutting their lines of action in a , b and g .

Then, taking moments about C,

$$P \cdot Ca + w \cdot Cg = W \cdot Cb;$$

$$\therefore P \cdot CA \cos \theta + w \cdot CG \cos \theta = W \cdot CB \cos \theta;$$

$$\therefore P \cdot CA + w \cdot CG = W \cdot CB,$$

which is independent of θ , and is therefore true for all values of θ .

Hence, the lever will be in equilibrium in any position

Example.—A uniform rod of 10 lbs. weight is hinged at one end and has a 10 lb. weight suspended from the other end. It is supported

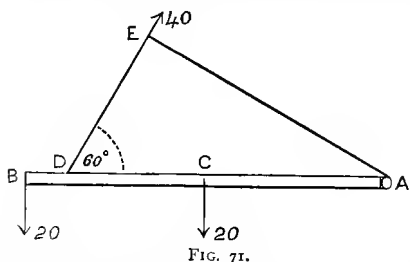


FIG. 71.

in a horizontal position by a string making an angle of 60° with the rod. Find the point to which the string must be attached when the tension of the string is = 40 lbs. weight.

Let AB be the rod hinged at A, C its middle point; let D be the

point to which the string is attached. Draw AE $\perp$ to the direction of the string, and take moments about A :

$$40 \times AE = 20 \times AB + 20 \times AC,$$

$$40 \times AD \sin 60^\circ = 20 \cdot AB + 10 \cdot AB,$$

$$40 \cdot \frac{\sqrt{3}}{2} \cdot AD = 30 \cdot AB;$$

$$\begin{aligned}\therefore AD &= \frac{30}{20\sqrt{3}} AB \\ &= \frac{\sqrt{3}}{2} AB.\end{aligned}$$

The principle of the lever is the principle which we use in the various machines for weighing heavy substances, such as the *Common* or *Roman Steelyard*, the *Danish Steelyard*, and the *Common Balance*.

The Common or Roman Steelyard.

108. This steelyard consists of a straight lever of the first class. One end of the lever is usually considerably heavier than the other, and the fulcrum or point of suspension is nearer to this heavy end.

The substance whose weight is required is attached to the extremity of the short arm by means of a hook, or by placing it upon a platform suspended from this end. Its weight is ascertained by sliding a movable weight along the longer arm until there is equilibrium, and noting the distance of the movable weight from the fulcrum when this is the case.

We will now proceed to graduate the steelyard, so that, by noting the graduation at which the movable weight rests, we should at once know the weight which is being supported.

Let AB be the steelyard, C the point of suspension. Let W be the weight to be weighed, which is attached to the end A ; P the movable weight attached to the ring at D . Let Q be the weight of the whole beam acting through G the centre of gravity.

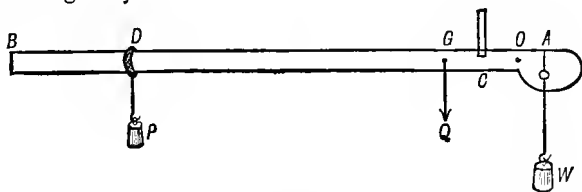


FIG. 72.

Now express the fact that the algebraical sum of the moments of the forces about C must be zero—

$$\therefore W \cdot AC = P \cdot DC + Q \cdot GC.$$

If G coincided with C, the moment $Q \cdot GC$ would vanish, and the equation of equilibrium would become

$$W \cdot AC = P \cdot DC,$$

or

$$\frac{W}{P} = \frac{DC}{AC}.$$

Hence W could be found in terms of P by measuring CD in terms of CA.

But if G does not coincide with C, mark off a distance CO along the rod such that the moment $Q \cdot GC =$ the moment $P \cdot CO$.

This can always be done, because P, Q, and CG are known quantities in any particular steelyard, and hence the distance CO must be known also.

Then the equation of equilibrium will become

$$\begin{aligned} W \cdot AC &= P \cdot DC + P \cdot CO \\ &= P (DC + CO) \\ &= P \cdot DO. \end{aligned}$$

Hence

$$\frac{W}{P} = \frac{DO}{AC}.$$

$$\begin{aligned} \text{If } DO &= AC, & W &= P; \\ \text{,, } DO &= 2 AC, & W &= 2 P; \\ \text{,, } DO &= n \cdot AC, & W &= n P. \end{aligned}$$

Hence O is taken as the zero point of graduation, and distances are marked off from O equal to AC, 2 AC, 3 AC, &c.; then, as P coincides with the 1st, 2nd, 3rd, &c., of these graduations, $W = P, 2 P,$ or $3 P,$ &c.

Note.—The student may find some difficulty in the *algebraical* treatment of the *mechanical* equation of equilibrium, where we change the sum of the moments $P \cdot DC + P \cdot CO$ into the moment $P \cdot DO$.

He may be helped by looking at the change from the following point of view :

We wish to get a relation between W and P only ; now

$$W \cdot AC = P \cdot DC + Q \cdot GC.$$

We then do away with the moment $Q \cdot GC$ by supposing that P's moment is taken about a point O, further from D than C is by

a distance CO , such that the additional moment $P.CO =$ the moment $Q.CG$.

If G lies on the right of C , Q will help W , and O the zero of graduation will be on the left of C .

If on account of the wearing away of the steelyard by constant use Q becomes diminished, the old graduations will no longer give the true weight.

Example.—In using a common steelyard whose weight has diminished by wear, and whose centre of gravity lies on the same side of the fulcrum as the weight to be weighed, a tradesman uses the old graduations; does he defraud his customers or himself?

Using the figure of the last article we have

$$W.AC + Q.GC = P.DC;$$

hence

$$W.AC = P.DC - Q.GC$$

$$= P.DO, \text{ when } P.CO = Q.GC, \text{ and } O$$

lies to the left of C .

If Q is decreased while G remains in the same position, CO must also be decreased; that is, O ought to be moved to the right. Hence every graduation ought to be moved to the right by the same distances; that is, the customer gets a little more than the true weight denoted by the graduation.

109. In most steelyards the weighing of heavy weights is accomplished by suspending small weights from the extremity of the long arm, and so avoiding the necessity of having a very long steelyard. Thus, in making a steelyard to weigh substances in stones, pounds, and fractions of a pound with a movable weight of 2 lbs., we might graduate the rod up to $7AC$, calling the consecutive graduations 2, 4 . . . 14, the last graduation denoting 7×2 lbs., that is, 14 lbs. or 1 stone.

Bisecting each of the graduations we should get the marks corresponding to 1, 3, 5 . . . 13 lbs.

We can then subdivide these graduations into whatever fractions of the whole would be convenient—say, quarters.

At the last graduation B a small platform is suspended on which we place multiples of 2 lbs.; each multiple of 2 lbs. which we place at B denoting an additional weight of 1 stone on the platform at A .

By means of this steelyard we can then weigh a substance in stones, pounds, and quarters of a pound.

The Danish Steelyard.

110. In this steelyard, which consists of a straight lever heavily weighted at one end, the substance to be weighed is suspended from the other end, the weight of the steelyard acting as the *power*.

The weight of the substance is determined by moving the point from which the steelyard is suspended to one side or the other until the steelyard rests in a horizontal position ; we then compare the lengths of the arms.

To graduate it.

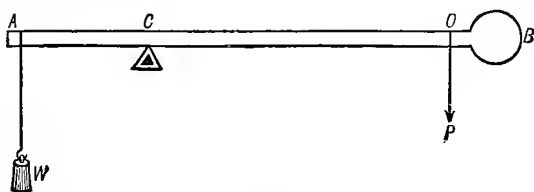


FIG. 73.

Take moments about C, the point of suspension :

$$P \cdot OC = W \cdot AC,$$

$$P(AO - AC) = W \cdot AC,$$

$$AC(P + W) = P \cdot AO, \quad (\text{See note.})$$

$$AC = \frac{P}{P + W} \cdot AO.$$

If $W = nP$,

$$AC = \frac{P}{P + nP} \cdot AO = \frac{AO}{n + 1}.$$

Now, put $n = 1, 2, 3, \&c.$,

$$AC = \frac{AO}{2}, \frac{AO}{3}, \frac{AO}{4}, \&c.,$$

which are the distances of the successive graduations from A.

Hence, to graduate this steelyard we measure distances from A equal to $\frac{1}{2} AO$, $\frac{1}{3} AO$, $\frac{1}{4} AO$, $\&c.$, and mark them 1, 2, 3, $\&c.$; and when the fulcrum coincides with the 1st, 2nd, 3rd, $\&c.$, of these graduations $W = P, 2P, 3P, \&c.$

Note.—This equation might have been obtained at once by taking moments about A, since the reaction of the fulcrum at $C = P + W$.

This steelyard is obviously not adapted for weighing heavy weights, for it is very tedious to move the steelyard one way or the other until it balances; and when the weight is a considerable multiple of P , the successive graduations approach very close to each other; thus the distance between the graduations 9 and 10 is only $\frac{A O}{110}$.

Again, in subdividing the space between successive graduations, each subdivision has to be calculated, since equal differences in weight do not produce equal distances between the corresponding graduations.

Thus, if C, C', C'' be the graduations for $7P, 7\frac{1}{2}P, 8P$,

$$AC = \frac{1}{8} AO, AC' = \frac{2}{17} AO, AC'' = \frac{1}{9} AO;$$

$$\therefore CC' = \frac{1}{136} \cdot AO, \text{ and } C'C'' = \frac{1}{153} \cdot AO.$$

Example.—A Danish steelyard weighs $1\frac{1}{2}$ lb. Graduate it to weigh ounces.

$$\text{Let } W = n \text{ oz., } P = 24 \text{ oz.,}$$

$$n \cdot AC = 24 OC,$$

$$n(AO - OC) = 24 OC,$$

$$(24 + n) OC = n \cdot AO;$$

$$\therefore OC = \frac{n}{24 + n} AO.$$

$$\text{Let } n = 1, \quad OC = \frac{1}{25} AO;$$

$$n = 2, \quad OC = \frac{2}{26} AO;$$

$$n = 3, \quad OC = \frac{3}{27} AO,$$

and so on.

These give us the successive graduations measured from O.

The Common Balance.

111. The balance consists of a lever AB suspended from C , such that its arms are perfectly symmetrical about the line CD , which bisects AB at right angles. From the ends A and B are suspended two scale-pans of equal size and weight, so that when the pans are empty the beam should be horizontal.

The substance to be weighed is placed in one of the pans, and known weights are placed in the other pan until the beam rests in a horizontal position.

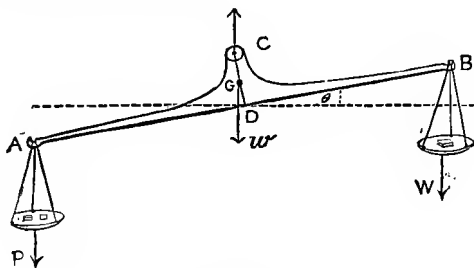


FIG. 74.

To determine the horizontality of the beam, a long needle is attached at right angles to the beam at D , which oscillates in front of a scale attached to the stand of the balance.

When the beam is in a horizontal position the weight W of the substance in the one pan ought to be equal to the weight P in the other.

If P and W differ by a little the beam would come to rest in a position inclined at an angle to the horizontal; we will now proceed to determine this angle θ for a given difference of P and W .

Suppose $P > W$.

Let w be the weight of the beam and scale-pans acting through the centre of gravity G ; the point G is always below C in the line CD , or CD produced.

Let $AB = 2a$ $CD = h$, $CG = k$.

To determine the position of equilibrium we shall take

moments about C ; we shall therefore require the perpendicular distances of C from the lines of action of P, W and w .

The perpendicular from D on the direction of P and W $= a \cos \theta$; the perpendicular from C on the vertical through D $= h \sin \theta$;

$$\therefore \text{the perpendicular from C on the direction of P} \\ = a \cos \theta - h \sin \theta ;$$

$$\therefore \text{the perpendicular from C on the direction of W} \\ = a \cos \theta + h \sin \theta ;$$

and the perpendicular from C on the direction of $w = k \sin \theta$;

$$\therefore P(a \cos \theta - h \sin \theta) = W(a \cos \theta + h \sin \theta) + w \cdot k \sin \theta ;$$

$$\therefore \sin \theta (W \cdot h + P \cdot h + w \cdot k) = \cos \theta (P \cdot a - W \cdot a) ;$$

$$\therefore \tan \theta = \frac{(P - W)a}{(P + W)h + w \cdot k}.$$

The quantities a , h , w , and k are always the same for the same balance ; the above equation therefore gives us the value of θ when P and W are known.

112. The **requisites for a good balance** are *sensibility* or *sensitiveness*, *stability* and *truth*.

By *sensible* we mean that when P and W differ by a very small amount the beam should show a distinct deflection from the horizontal ; that is, for a given difference of P and W, θ should be large.

$$\text{Now} \quad \tan \theta = \frac{a}{(P + W)h + w \cdot k} \cdot (P - W) ;$$

$$\therefore \frac{a}{(P + W)h + w \cdot k} \text{ is a measure of the sensibility.}$$

Consequently a should be large in comparison with h and k , and w should be small.

It is not convenient to have the arms of the balance very long, but h and k can be made as small as we please, and for very accurate weighing w is made as small as possible, due regard being paid to the strength of the balance.

By *stable* we mean that when the balance is slightly displaced from its position of equilibrium it should tend to return quickly.

For this it is necessary that h and k should be as large as possible ; hence the requirements for stability are at variance with the requirements for sensibility.

Thus in making a balance we have to take into account the uses to which it will be put in the future ; whether great accuracy or speed in weighing is most essential.

Thus in making a chemical balance for use in the laboratory we pay special attention to the sensibility without neglecting stability ; while for a balance required for heavy goods, where extreme accuracy is not of great importance, stability is the chief requisite.

By *true* we mean that the weights should be equal when the beam is horizontal.

This is a necessary requisite for all balances, and is easily tested by interchanging the weights P and W ; if the balance does not remain in a horizontal position, P and W are not equal and the balance is *false*.

This is probably due to the fact that one arm is longer than the other, but so weighted that the balance is horizontal when the pans are empty ; but when weights are placed in the pans, the larger weight suspended from the shorter arm will be balanced by the smaller weight suspended from the longer arm.

113. Proposition. To determine the true weight of a body in a false balance.

Let W be the true weight of the substance which is required. Weigh the substance first in one pan and then in the other. Let P and Q be the apparent weights ; let x, y be the lengths of the arms.

Take moments about C , the point of suspension:

$$W \cdot x = P \cdot y ; W \cdot y = Q \cdot x.$$

Eliminate x and y by multiplying the equations together and dividing both sides of the resulting equation by the product xy .

Then

$$\begin{aligned} W^2 &= P \cdot Q ; \\ \therefore W &= \sqrt{P \cdot Q}. \end{aligned}$$

Hence the true weight is the geometric mean between the apparent weights.

If we eliminate W by division, we get

$$\frac{x}{y} = \frac{P \cdot y}{Q \cdot x};$$

$$\therefore \frac{x}{y} = \frac{\sqrt{P}}{\sqrt{Q}},$$

which gives us the ratio of the arms.

114. If the falseness of a balance is not due to the unequal lengths of the arms, it may be due to the fact that one of the pans is *loaded*, in which case the balance would not be horizontal when the pans are empty.

In this case, if we weighed the substance W in both pans, and if the apparent weights were P and Q , and if w were the weight of the *loading* given to one pan, we should have

$$W + w = P$$

$$W = Q + w.$$

Eliminate w ,

$$\therefore W = \frac{P + Q}{2}.$$

So, if we eliminate W we shall get the amount of the loading,

$$w = \frac{P - Q}{2}.$$

Example.—A grocer who knows that his balance is false weighs his customer $\frac{1}{2}$ lb. of tea out of one pan and $\frac{1}{2}$ lb. out of the other. Does the customer get more or less than 1 lb.?

Let x and y be the lengths of the arms, P and Q the weights he actually gets.

Then $P \cdot x = \frac{1}{2} \cdot y$, and $Q \cdot y = \frac{1}{2} \cdot x$;

$$\therefore P = \frac{y}{2x} \text{ lb.}, \quad Q = \frac{x}{2y} \text{ lb.};$$

$$\therefore P + Q = \frac{y}{2x} + \frac{x}{2y} = \frac{x^2 + y^2}{2xy} \text{ lb.}$$

Now $x^2 + y^2 > 2xy$, because $x^2 + y^2 - 2xy = (x - y)^2$, which is positive;

$$\therefore P + Q > 1 \text{ lb.}$$

This result could also have been obtained at once from the result of Article 113, since the arithmetic mean between two quantities is greater than their geometric mean.

The Wheel and Axle.

115. This machine, which is a simple adaptation of the principle of the lever, consists of a long cylinder—the *Axle*—whose axis ends in two pivots resting in fixed sockets.

Rigidly attached to this cylinder is a larger cylinder of no great thickness—the *Wheel*—having the same axis.

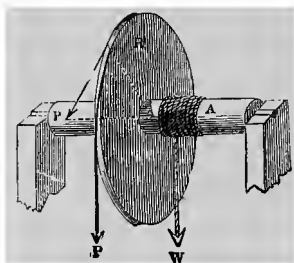


FIG. 75.

The power is applied to a rope which is wrapped round the circumference of the wheel, and the weight is attached to a rope which is wrapped in the opposite direction round the axle.

Thus, when we apply the power we turn the wheel, which also turns the axle, and thereby raises the weight.

This machine is often used for raising a bucket of water from a well, where the power is applied by the hand to a handle which takes the place of the wheel.

The capstan for raising the anchor on board ship is another example of the same machine, where the power is applied at the ends of long spokes fitted into the axle.

116. **To determine the condition of equilibrium.**—Let the figure represent a cross section of the machine by a vertical plane perpendicular to the common axis, and cutting it at C.

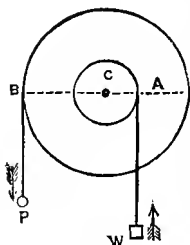


FIG. 76.

Let A and B be the points where the ropes leave axle and wheel respectively, their directions here being tangential to the wheel and axle, so that ACB is perpendicular to P and W.

Take moments about the common axis—that is, about C.

$$P \cdot CB = W \cdot CA;$$

$$P \times (\text{radius of wheel}) = W \times (\text{radius of axle}).$$

$$\text{Efficiency} = \frac{\text{radius of wheel}}{\text{radius of axle}}.$$

117. A greater mechanical advantage may be obtained by means of the *Differential Wheel and Axle*, in which the weight is attached to a pulley which hangs in the loop of a string, one end of which is wrapped round the axle as before, and the other end is wrapped in the opposite direction round an axle of smaller radius projecting out of the main axle.

Let T be the tension of the string round the pulley.

Then $2 T = W$. (See Art. 119.)

Let the radii of the power arm and the two axles be a , b , c respectively.

Taking moments about the common axis we have :

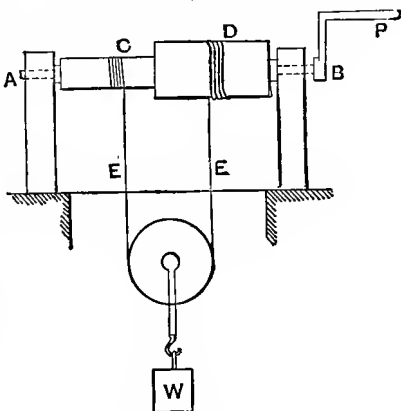


FIG. 76a.

$$P \cdot a = T \cdot b - T \cdot c,$$

$$P \cdot a = T (b - c)$$

$$= \frac{W}{2} (b - c);$$

$$\therefore \frac{W}{P} = \frac{2a}{b - c}.$$

Now $b - c$ may be made as small as we please by making the radii of the axles nearly equal; hence the mechanical advantage may be increased indefinitely.

The Toothed or Cogged Wheel.

118*.—A chapter on machines would be incomplete if we omitted to mention the cogged wheel, which is constantly employed in such things as clocks, watches, and bicycles, and frequently in combination with the wheel and axle, as in cranes.

Two wheels of different radii are made with teeth or cogs of equal size cut on the outside of their circumferences. They are then placed in the machine so that the tooth of one wheel fits between two consecutive teeth of the other, the two wheels revolving in the same plane in opposite directions.

If P and W act at right angles to the radii of the two wheels there is no mechanical advantage, but if W is attached to an axle fixed to the small wheel then, if a, b, c be the radii of the large

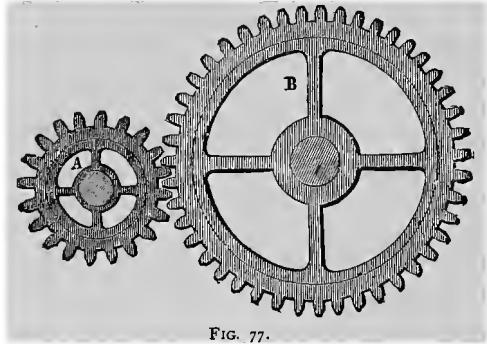


FIG. 77.

wheel, small wheel, and axle respectively, and if F be the mutual action between the cogs, we have :

$$P \cdot a = F \cdot a,$$

$$W \cdot c = F \cdot b ;$$

$$\therefore \frac{P}{W} = \frac{c}{b}.$$

The chief use made of the cogged wheel is due to the difference in the velocities of revolution of the two wheels owing to their difference in radius, for their velocities of revolution vary inversely as the number of cogs on their circumference—that is, inversely as their radii.

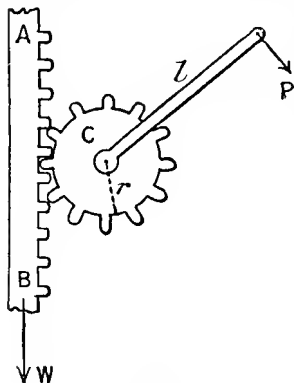


FIG. 78.

A modification may be seen in the apparatus for raising the gates of sluices, in which case the cogs of the wheel fit into the spaces of a cogged vertical bar which carries the gate.

This is called the Rack and Pinion.

For the conditions of equilibrium take moments about C :

$$W \times r = P \times l,$$

where r is the radius of the cogged wheel, and l the length of the arm at which P acts.

The Pulley.

119. The *pulley* is a small circular disc of wood or metal, movable about an axis through its centre, the ends of which axis rest in sockets in the framework of the *pulley-block*.

A groove is cut in the outer edge of this disc, so as to keep in position the rope or cord which is passed round it. A hook is fastened to one end of the pulley-block for the purpose of supporting the pulley or attaching a weight to it.

The power is applied at one end of the string, which passes round a portion of the pulley, the other end being attached to the weight or to a fixed support, or sometimes (in combinations of pulleys) to another pulley.

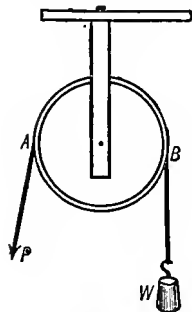


FIG. 79.

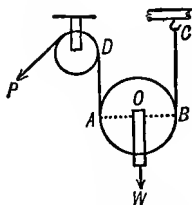


FIG. 80.

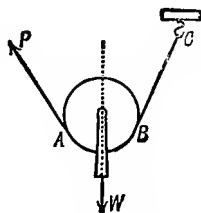


FIG. 81.

If the pulley-block is attached to a fixed support the pulley is said to be *fixed*, as in fig. 79; otherwise it is said to be *movable*, as in fig. 81. In fig. 80 one pulley is *fixed*, the other *movable*.

We suppose that the pulley is perfectly smooth, so that the tension of the string is the same on one side of the pulley as it is on the other.

A fixed pulley or smooth peg is often used to change the direction of a force, without altering its effect on the weight, as in fig. 79.

Here the tension of the string is the same throughout,

$$\therefore P = W.$$

In fig. 80 we have a single movable pulley to which the weight is attached ; the pulley is supported in the loop of a string, at one end of which the power is applied, the other end being attached to a fixed support the two portions of the string A D, B C being parallel.

The tension all along the string is $= P$.

Hence W acting vertically downwards is balanced by two tensions, each $= P$, acting vertically upwards ;

$$\therefore W = 2 P.$$

If the weight of the pulley be w ,

$$W + w = 2 P.$$

The fixed pulley on the left merely changes the final direction of P .

In fig. 81 we also have a single movable pulley, but the portions of the string on either side of it are not parallel. Since the tension of each portion is $= P$, they must make equal angles with the direction of the third force W , *i.e.* with the vertical. Calling this angle θ , and resolving the forces vertically, we have,

$$\begin{aligned} W + w &= P \cos \theta + P \cos \theta \\ &= 2 P \cos \theta. \end{aligned}$$

Systems of Pulleys.

120. There are three so-called systems or combinations of pulleys usually given in statics.

The First System is that in which each pulley hangs in the loop of a separate string, one end of which is attached to a fixed beam or support, and the other end to the pulley next above it, and so on, the power acting at the other end of the string which passes round the highest pulley.

The Second System consists of two blocks of pulleys, in which the pulleys are placed side by side and movable about a common axis, or else they are placed one underneath the other.

The upper block is attached to a fixed support, the weight being attached to the lower block ; the same string passes round each pulley, beginning with the upper block, when

the power acts downwards. The power acts at one end of this string, the other end is finally attached to the upper or lower block. If the power acts upwards, the string must first pass round one of the pulleys in the lower block.

The Third System is that in which a string passes over each pulley, one end being attached to the weight and the other to the pulley next below it, and so on, the power acting at the other end of the string which passes over the lowest pulley ; the top pulley being attached to a fixed support.

(See the figures on the following pages.)

Note.—In considering the conditions of equilibrium, we shall suppose that all the strings are parallel and vertical.

121. Condition of equilibrium in the first system of pulleys.

This is the system in which each pulley hangs in the loop of a separate string.

Let A, B, C, &c., be the pulleys, and let t_1, t_2, t_3 , &c., denote the tensions of the strings which pass round the pulleys C, B, A, &c., respectively.

The pulley on the right merely alters the direction of the last string.

(1) When the weights of the pulleys are neglected.

For the equilibrium of the pulley C we

have $W = 2 t_1 ; \therefore t_1 = \frac{W}{2}$.

For the equilibrium of the pulley B we have

$$t_1 = 2 t_2 ; \therefore t_2 = \frac{t_1}{2} = \frac{W}{4}.$$

For the equilibrium of the pulley A we have

$$t_2 = 2 t_3 ; \therefore t_3 = \frac{t_2}{2} = \frac{W}{8}.$$

Thus, if there are only three movable pulleys, we have

$$W = 8 t_3 = 2^3 \cdot t_3 ;$$

but

$$t_3 = P ; \therefore W = 2^3 \cdot P.$$

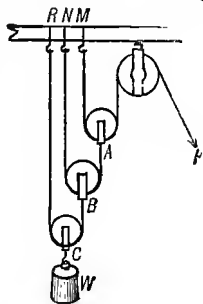


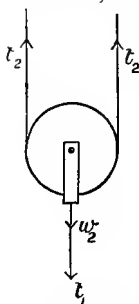
FIG. 82.

Similarly, if there were four movable pulleys, $W = 2^4 \cdot P$, and so on; and if there were n movable pulleys, $W = 2^n \cdot P$.

(2) When the weights of the pulleys are taken into consideration.

Let $w_1, w_2, w_3 \dots w_n$ be the weights of the n pulleys C, B, A, &c.

Then, considering the equilibrium of each pulley separately, as in the accompanying figure, and beginning with the lowest pulley, we have the following equations:



$$2 t_1 = W + w_1 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$2 t_2 = t_1 + w_2 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$2 t_3 = t_2 + w_3 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and so on;

$$2 t_n = t_{n-1} + w_n \quad . \quad . \quad . \quad . \quad . \quad (n)$$

and $t_n = P$.

FIG. 82a.

If we now multiply the equations (2), (3) . . . (n) by $2, 2^2, 2^3 \dots 2^{n-1}$ we get the following:

$$2 t_1 = W + w_1,$$

$$2^2 t_2 = 2 t_1 + 2 w_2,$$

$$2^3 t_3 = 2^2 t_2 + 2^2 w_3,$$

and so on;

$$2^n P = 2^{n-1} t_{n-1} + 2^{n-1} w_n.$$

Now add the corresponding sides of the equations together, and cancel out similar terms on both sides of the resulting equation, and we get

$$2^n \cdot P = W + w_1 + 2 w_2 + 2^2 w_3 + \dots + 2^{n-1} w_n;$$

$$\text{or} \quad P = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{2 w_2}{2^{n-1}} + \frac{2^2 w_3}{2^{n-2}} + \dots + \frac{w_n}{2}.$$

If the weights of all the pulleys are equal, and each = w , we have,

$$\begin{aligned} 2^n P &= W + w + 2 w + 2^2 w + \dots + 2^{n-1} w \\ &= W + w (2^n - 1) \text{ by geometrical progression.} \end{aligned}$$

Note.—The resultant force on the beam (omitting the right-hand pulley)

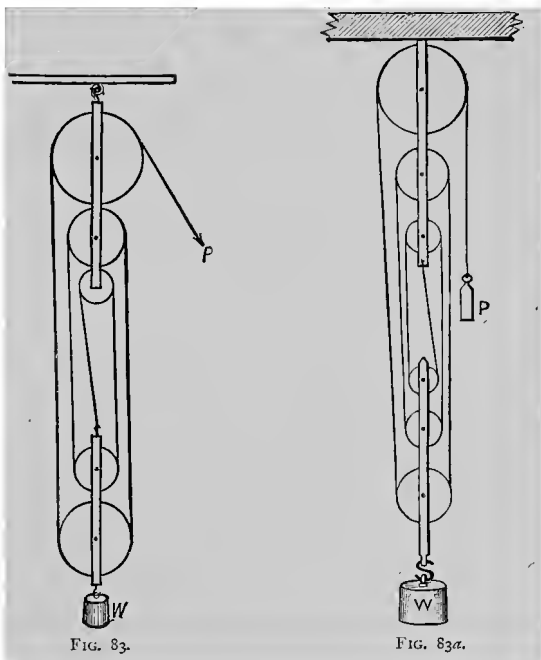
$$= t_1 + t_2 + t_3 + \dots$$

If we call this force R ,

$$R + P = W + w_1 + w_2 + \dots + w_n.$$

122. Condition of equilibrium in the second system of pulleys.

This is the system in which there are two blocks and the same string passes round each pulley.



If the string is fastened to the upper block, the number of pulleys in each block will be the same; but if the string is fastened to the lower block, there must be one more pulley in the upper block than in the lower block.

When P acts downwards, the two blocks are connected by the various portions of the string just as many times as there are pulleys. Now the tension of the string is the same throughout and $= P$; hence, if there be n pulleys, the resultant upward tension on the lower block $= nP$; hence, if W be the weight supported, and w the weight of the lower block,

$$nP = W + w.$$

Note.—The resultant force on the beam

$$= W + P + \text{the weights of the two blocks.}$$

If P were to act upwards, the top pulley in the above figures would be unnecessary, and the conditions would be slightly altered.

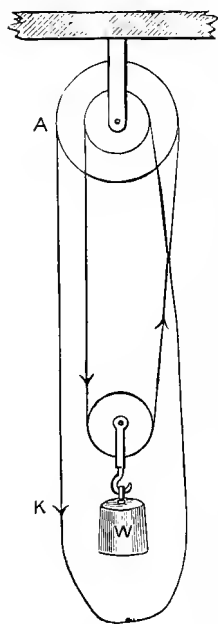


FIG. 84.

A modification of the differential wheel and axle, called the *Differential Pulley* or the *American Block*, is of very great practical value.

The upper block consists of two pulleys of different radii movable about a common axis; the lower block consists of a single movable pulley, to which the weight is attached.

An endless chain passes over these two upper pulleys and under the lower one; this chain cannot slip over the upper pulleys on account of their being made with projections in their rims which fit the links of the chain; consequently, W hangs in equilibrium without any power being applied if the upper pulleys are nearly equal.

To raise W , P is applied to the left-hand portion of the chain.

Let T be the tension of the portion of chain between the bottom pulley and the upper block, and let b and c be the radii of the pulleys in the upper block; then, neglecting the weights of the bottom pulley and the chain, we have

$$2T = W, \text{ for the equilibrium of the bottom pulley,}$$

and $P \cdot b + T \cdot c = T \cdot b$, for the equilibrium of the upper block;

$$\therefore P \cdot b = T (b - c) = \frac{W}{2} (b - c),$$

$$\frac{P}{W} = \frac{b - c}{2b}.$$

123. Condition of equilibrium in the third system of pulleys.

This is the system in which each string is attached to the weight.

Let A, B, C, &c., be the n pulleys, and let $t_1, t_2, t_3 \dots t_n$ denote the tensions of the strings which pass round them respectively.

(1) When we neglect the weights of the pulleys we have the following equations :

$$t_1 = P.$$

For the equilibrium of A, $t_2 = 2 t_1$
 $= 2 P,$

„ „ B, $t_3 = 2 t_2$
 $= 2^2 P,$

and so on.

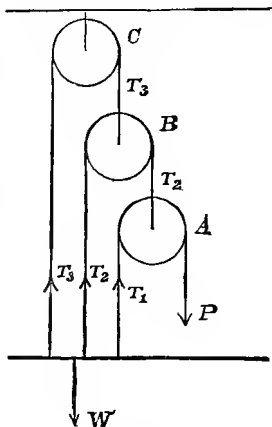


FIG. 85.

For the equilibrium of the n^{th} pulley $t_n = 2 t_{n-1} = 2^{n-1} \cdot P.$

Now W is supported by all the tensions ;

$$\begin{aligned} \therefore W &= t_1 + t_2 + t_3 + \dots + t_n \\ &= P + 2 P + 2^2 P + \dots + 2^{n-1} P \\ &= P (2^n - 1). \end{aligned}$$

(2) When the weights of the pulleys are taken into consideration.

Let the weights of A, B, C, &c., be $w_1, w_2, w_3 \dots w_{n-1}$, the weight of the n^{th} pulley not being taken into account since it is attached to a fixed beam and is not supported by any of the strings.

We then get the following equations :

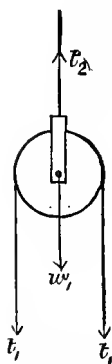


FIG. 85''.

$$\begin{aligned}
 t_1 &= P, \\
 t_2 &= 2 t_1 + w_1 = 2 P + w_1, \\
 t_3 &= 2 t_2 + w_2 = 2^2 P + 2 w_1 + w_2, \\
 \text{and so on; } &\dots \\
 t_n &= 2 t_{n-1} + w_{n-1} \\
 &= 2^{n-1} P + 2^{n-2} w_1 + 2^{n-3} w_2 + \dots w_{n-1}; \\
 \therefore W &= t_1 + t_2 + t_3 + \dots t_n \\
 &= P \\
 &\quad + 2 P + w_1 \\
 &\quad + 2^2 P + 2 w_1 + w_2 \\
 &\quad + 2^3 P + 2^2 w_1 + 2 w_2 + w_3 \\
 &\quad + \&c.
 \end{aligned}$$

$$\begin{aligned}
 &+ 2^{n-1} P + 2^{n-2} w_1 + 2^{n-3} w_2 + \dots + w_{n-1} \\
 &= P (2^n - 1) + w_1 (2^{n-1} - 1) + w_2 (2^{n-2} - 1) \\
 &\quad + \dots + (2 - 1) w_{n-1}.
 \end{aligned}$$

If the weights of the pulleys are all equal and each = w we have :

$$\begin{aligned}
 W &= P (2^n - 1) + w [2^{n-1} + 2^{n-2} + \dots + 2 - (n - 1)] \\
 &= P (2^n - 1) + w \{2(2^{n-1} - 1) - n + 1\} \\
 &= P (2^n - 1) + w \{2^n - 2 - n + 1\} \\
 &= P (2^n - 1) + w \{2^n - n - 1\}.
 \end{aligned}$$

In this system when the weight is suspended from a bar it must be attached to the point in the bar at which the resultant of all the tensions $t_1, t_2, t_3 \dots$ acts, otherwise the bar would not be in equilibrium, and would tilt up to one side or the other.

In general, the weight should be suspended from a point in the bar between the ends of the two last strings.

Note.—In the third system the weights of the pulleys assist P in supporting the weight, whereas in the first and second systems P has to support them as well as W . Theoretically, then, the third system would be the most advantageous to use ;

practically, we rarely see either the first or third in use with more than two movable pulleys, since the pulleys soon come in contact with the beam unless the strings are made of very great length.

The second system is the one commonly found in use.

Example 1.—In the first system of pulleys there are four movable pulleys each weighing 1 lb. weight; what weight can a power of 2 lbs. weight support?

With the usual notation let t_1, t_2, t_3, t_4 be the tensions of the strings, commencing with the lowest pulley.

$$\begin{aligned}
 \text{Then} \quad t_4 &= 2, \\
 t_3 + 1 &= 2 t_4, & \therefore t_3 &= 3; \\
 t_2 + 1 &= 2 t_3, & \therefore t_2 &= 5; \\
 t_1 + 1 &= 2 t_2, & \therefore t_1 &= 9; \\
 W + 1 &= 2 t_1, & \therefore W &= 17; \\
 & \therefore W = 17 \text{ lbs. weight.}
 \end{aligned}$$

Example 2.—In the first system a power of 3 lbs. weight balances a weight of 10 lbs. when there are three movable pulleys; find the weight of each.

Let w = the weight of each pulley.

Then, with the usual notation

$$\begin{aligned}
 2 t_1 &= 10 + w, & \therefore t_1 &= 5 + \frac{w}{2}; \\
 2 t_2 &= t_1 + w \\
 &= 5 + \frac{3w}{2}, & \therefore t_2 &= \frac{5}{2} + \frac{3w}{4}; \\
 2 t_3 &= t_2 + w \\
 &= \frac{5}{2} + \frac{7w}{4}, & \therefore t_3 &= \frac{5}{4} + \frac{7w}{8}. \\
 \text{But} \quad t_3 &= P = 3; & \therefore 3 &= \frac{5}{4} + \frac{7w}{8}; \\
 & \therefore 7w = 24 - 10 = 14; \\
 & \therefore w = 2 \text{ lbs. weight.}
 \end{aligned}$$

Note.—In the first system of pulleys, if W is given and P is required, begin with the lowest pulley and work up; but if P is given and W is required, begin with the top pulley and work down.

In all examples where the weights of the pulleys are given the student is advised never to use the formulæ which have been obtained in the previous articles.

Example 3.—In the second system a power of 5 lbs. weight can support a weight of 23 lbs., the weight of the lower block being 7 lbs. What power can support a weight of 41 lbs.?

Let n = the number of strings.

Then $n P = W + w,$

$$n \cdot 5 = 23 + 7$$

$$= 30 ;$$

$$\therefore n = 6.$$

When $W = 41,$

$$6 P = 41 + 7$$

$$= 48 ;$$

$$\therefore P = 8 \text{ lbs. weight.}$$

Example 4.—In the third system, a power of 3 lbs. weight can support a weight of 67 lbs. when there are four pulleys of equal weight. Find the weight of each pulley.

Let w = the weight of each pulley.

Then, with the notation adopted in Art. 123,

$$t_1 = P = 3,$$

$$t_2 = 2 t_1 + w = 6 + w,$$

$$t_3 = 2 t_2 + w = 12 + 3 w,$$

$$t_4 = 2 t_3 + w = 24 + 7 w,$$

$$67 = t_1 + t_2 + t_3 + t_4 = 45 + 11 w ;$$

$$\therefore 11 w = 22 ;$$

$$\therefore w = 2.$$

The Inclined Plane.

124. The *Inclined Plane* is usually a slanting rigid board inclined to the horizon at an angle less than a right angle, which angle is called the *angle of the plane*. It may frequently be seen in use in getting barrels out of cellars, or in transferring luggage between a wharf and the deck of a ship.

Since the inclined plane itself supports a portion of the weight, the power required to keep the weight in equilibrium will usually be less than the weight.

To find the condition of Equilibrium.

(1) When the power acts up the plane and parallel to it.

Let the figure represent a section of the inclined plane by a vertical plane through the weight and perpendicular to the edge of the plane.

Let P be the power, W the weight, R the pressure of the plane which is perpendicular to the plane, and let i be the angle of the plane.

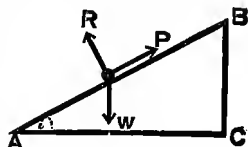


FIG. 86.

Then, by Lami's theorem,

$$\begin{aligned} P : W : R &:: \sin (W, R) : \sin (R, P) : \sin (P, W) \\ &:: \sin (180^\circ - i) : \sin (90^\circ) : \sin (90^\circ + i) \\ &:: \sin i : 1 : \cos i ; \end{aligned}$$

$$\therefore P = W \sin i, \text{ and } R = W \cos i.$$

These results might also have been obtained by resolving the forces along AB and $\perp$ to AB , or by applying the triangle of forces directly.

Since $P = W \sin i$, and $\sin i$ is always < 1 , P must be $< W$.

Thus there is always mechanical advantage.

(2) Where the power acts horizontally.

Using the same notation we have, by Lami's theorem,

$$\begin{aligned} P : W : R &:: \sin (W, R) : \sin (R, P) \\ &:: \sin (P, W) \\ &:: \sin (180^\circ - i) \\ &:: \sin (90^\circ + i) : \sin (90^\circ) \\ &:: \sin i : \cos i : 1 ; \\ \therefore P &= W \tan i, R = W \sec i. \end{aligned}$$

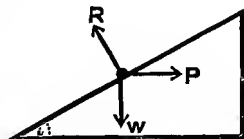


FIG. 87.

Here there is only mechanical advantage so long as $\tan \alpha < 1$; that is, so long as $\alpha < 45^\circ$.

(3) When P acts at an angle θ with the plane.

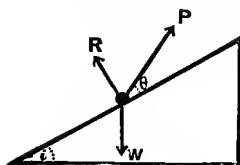


FIG. 88.

We have as before,

$$\begin{aligned} P : W : R &:: \sin(W, R) : \sin(R, P) \\ &:: \sin(P, W), \\ &:: \sin(180-i) : \sin(90-\theta) \\ &:: \sin(90+i+\theta), \\ &:: \sin i : \cos \theta : \cos(i+\theta). \end{aligned}$$

The results of cases (1) and (2) would be obtained from this by putting $\theta = 0$ and $-i$ respectively.

Example 1.—Two weights W, W' are resting on two inclined planes which have a common vertex, the weights being connected by a string passing over a pulley at the top. Find the relation between the weights, the angles of the planes being i, i' .

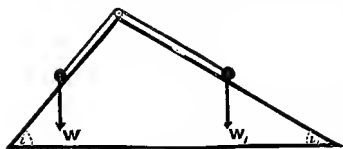


FIG. 89.

Let T be the tension of the string.

Then

$$\begin{aligned} \frac{T}{W} &= \sin i, \quad \frac{T}{W'} = \sin i'; \\ \therefore W \sin i &= W' \sin i'; \\ \therefore \frac{W}{W'} &= \frac{\sin i'}{\sin i}. \end{aligned}$$

If l, l' be the lengths of the planes, and h the common altitude,

$$\begin{aligned} \sin i &= \frac{h}{l}, \quad \sin i' = \frac{h}{l'}; \\ \therefore \frac{W}{W'} &= \frac{\frac{h}{l'}}{\frac{h}{l}} = \frac{l}{l'}. \end{aligned}$$

Example 2.—On a certain inclined plane the power required to support a weight when acting parallel to the plane is half that which will support the same weight when it acts horizontally. Find the angle of the plane.

Let α = the angle of the plane,

P the power acting up the plane which will support W,

2P " " horizontally " "

Then $\frac{P}{W} = \sin \alpha, \quad \frac{2P}{W} = \tan \alpha;$

$$\therefore 2 \sin \alpha = \tan \alpha,$$

or $2 \sin \alpha = \frac{\sin \alpha}{\cos \alpha},$

or $2 \sin \alpha \cos \alpha = \sin \alpha.$

Now $\sin \alpha$ cannot = 0 ; therefore, dividing by $\sin \alpha$, we have

$$2 \cos \alpha = 1,$$

or $\cos \alpha = \frac{1}{2};$

$$\therefore \alpha = 60^\circ.$$

The Screw.

125. The Screw in its simplest form consists of a cylinder of wood or metal with a projecting spiral thread running round its circumference, which thread always makes the same angle with all lines parallel to the axis of the cylinder.

We may imagine it to be constructed in the following manner :

Let $A a d D$ be a rectangle whose base $A a$ is equal to the circumference of the cylinder. In the sides $A D, a d$ mark off $A B, B C, C D, a b, b c, c d$, all equal to the vertical distance between two consecutive revolutions of the spiral thread.

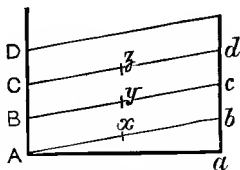


FIG. 90

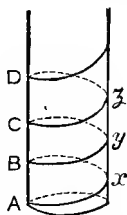


FIG. 91.

Join $A b, B c, C d$, &c. Now apply the rectangle to the surface of the cylinder, so that $A a$ coincides with the base of

the cylinder, the points a, b, c, d , &c., coinciding with the points A, B, C, D , &c., respectively ; the lines $A b, B c, C d$ will now form a continuous spiral line on the surface of the cylinder, their middle points x, y, z becoming the points x, y, z on the cylinder. If we now suppose this line to project from the surface of the cylinder we shall have the *thread* of the screw.

When the cylinder is vertical the angle which the thread makes with the horizontal is constant and is called the *pitch* of the screw ; it is, of course, the complement of the constant angle which the thread makes with lines parallel to the axis of the cylinder. It is the angle $a A b$.

If we call this angle α ,

$$\tan \alpha = \frac{a b}{A a} = \frac{\text{vertical distance between two threads}}{\text{circumference of the cylinder}}.$$

Let a be the radius of the cylinder, then circumference of the cylinder $= 2 \pi a$;

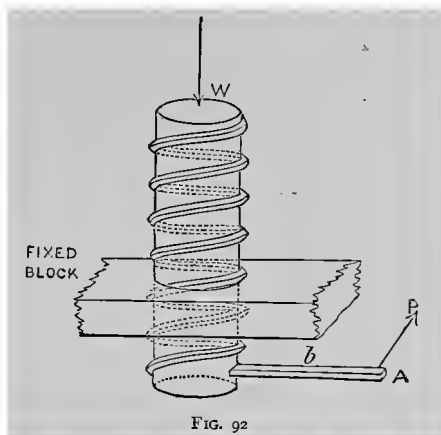
$$\therefore \text{vertical distance between two threads} = 2 \pi a \cdot \tan \alpha.$$

This cylinder with its projecting thread works inside a hollow cylinder of equal radius, in which there is a spiral groove of the same *pitch* and the same shape as the thread on the cylinder.

This hollow cylinder and groove are cut in a block, which is usually fixed. The solid cylinder and the hollow cylinder into which it fits are called *companion* screws. When in working, the weight or resistance which has to be overcome is at the end of the screw ; the power is applied in a plane $\perp^r$ to the axis, at right angles to an *arm* which is also $\perp^r$ to the axis ; since the screw can only move through its *companion* in the fixed block in the direction of the axis, this is the motion which the power produces, and so the resistance at the end of the screw is overcome.

Some idea of the mode of action will be conveyed by the following figure.

126. The thread of the screw and the groove of the companion screw may be of any shape—in considering the



condition of equilibrium we shall take it to be rectangular—the mutual pressure between the surfaces of the thread of the screw and the groove being perpendicular to these surfaces; thus the portion of the thread which at any time is within the groove will be of the nature of a *continuous inclined plane* whose inclination to the horizon is α .

Thus, if O be any point of the thread of the screw inside the groove, w that portion of the weight which is supported at O, p the horizontal force at O which is due to the force P at A, and r the mutual pressure at O between the screw and its companion $\perp r$ to the surface at O, we have the second case of the inclined plane (Art. 124),

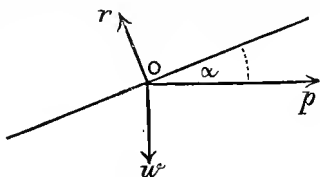


FIG. 93.

$$p : w : r :: \sin \alpha : \cos \alpha : 1.$$

Thus $p = w \cdot \tan \alpha$ at every point of the thread;

$$\therefore \Sigma(p) = \Sigma(w \tan \alpha) = \tan \alpha \cdot \Sigma(w), \text{ since } \alpha \text{ is constant} \\ = \tan \alpha \cdot W \quad (1)$$

Further, the sum of the moments of all the forces p about the axis is = the moment of P ; therefore if b = the distance of A from the axis,

$$\begin{aligned} P \cdot b &= \Sigma (p \cdot a) \\ &= a \cdot \Sigma (p), \text{ since } a \text{ is constant} \\ &= a \tan \alpha \cdot W, \text{ by equation (1) ;} \end{aligned}$$

$$\begin{aligned} \therefore \frac{P}{W} &= \frac{a \tan \alpha}{b} \\ &= \frac{2 \pi a \tan \alpha}{2 \pi b} \end{aligned}$$

$$= \frac{\text{vertical distance between two consecutive revolutions of the thread}}{\text{circumference of the circle whose radius is } b}.$$

Theoretically there is no limit to the mechanical advantage which can be gained by the screw, since P 's arm can be made as large as we please, and the distance between two threads can be made as small as we please ; yet, in practice, it is not convenient to have P 's arm very long, nor is it advisable to have consecutive turns of the thread very close together, since the strength of the screw is thereby considerably impaired.

It is largely made use of in cases where large weights have to be moved slowly forward, as in the case of raising the roof of a building.

The Wedge.

127. The Wedge is a solid triangular prism which is used for the purpose of splitting a body into two parts.

The edge of the wedge is introduced between the particles of the substance at the required place, and is then thrust forward by the application of some force, as the blow of a hammer ; by such means a very great pressure is produced, which is capable of tearing asunder the particles of substances which adhere to each other most closely.

We will suppose the wedge isosceles, so that it presents the appearance of two equal inclined planes placed base to base.

In the figure we may suppose the wedge to be used for splitting the trunk of a tree.

Let R, R' be the pressures at the points of contact of the wedge with portions of the trunk; let 2α be the angle of the wedge, so that each face makes an angle α with the vertical; and let P be the force applied at the top.

Resolving the forces vertically and horizontally, we have

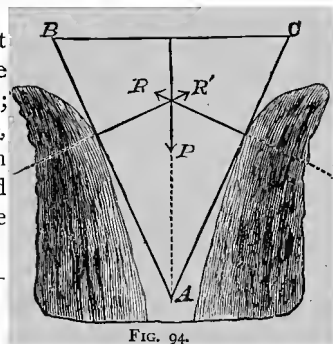


FIG. 94.

$$P = R \sin \alpha + R' \sin \alpha \quad . \quad . \quad . \quad (1)$$

$$R \cos \alpha = R' \cos \alpha \quad . \quad . \quad . \quad (2)$$

From (2)

$$R = R';$$

$$\therefore P = 2R \sin \alpha.$$

It is obvious that, if P is constant, R increases as α decreases; and when P is great and α is small R can be made very great indeed.

Besides the wedge in its ordinary well-known form most of our cutting tools are modifications of it; thus knives, axes, and chisels are all modifications of the wedge.

128. It is scarcely necessary to remind the student that in the practical use of machinery the power employed has not only to counterbalance the resistance, but to overcome it—that is to say, work has to be done. Hence in all cases the power required to work the machine must be greater than that obtained in the investigations of this chapter; and if the power applied be sufficiently great to overcome the resistance of the *weight* and the friction of the various parts of the machine, and to set the mass of the machinery in motion, the machine will continue to work at an ever-increasing rate of speed (by the second law of motion); and as soon as the required rate has been attained the power may then be decreased so as just

to counterbalance or be in equilibrium with these various forces, when the machine will continue to work with uniform speed so long as this power is kept up.

Examples.

The Lever.

1. 3 lbs. and 7 lbs. balance each other on a lever of the first-class whose length is 1 foot. Where is the fulcrum?

2. 5 lbs. and 8 lbs. balance each other on a lever whose short arm is 7 ins. What is the length of the lever?

3. 3 lbs. and 4 lbs. balance on a lever whose long arm is 9 ins. What is the length of the other arm?

4. A uniform rod of 4 lbs. weight and 1 foot long has 6 lbs. and 8 lbs. suspended from its ends. About what point will it balance?

5. Find the length of a lever of the second class in order that a force of 5 lbs. weight may balance a weight of 10 lbs., their points of application being 1 foot apart.

6. A uniform beam 10 feet long of 140 lbs. weight turns freely about a point 4 feet from one end, and at that end a weight of 41 lbs. is hung. What weight must be hung from the other end to keep it in equilibrium?

7. A uniform beam 8 feet long of 24 lbs. weight turns freely about one end and is kept in a horizontal position by a force of 24 lbs. weight at the other end. At what angle must this force act?

8. Masses of 7 lbs. and 20 lbs. are hung at the ends of a straight rod whose length is 2 ft. 8 ins. and whose weight is 25 lbs. About what point will the rod balance?

9. The arms of a straight lever are in the ratio 1 : 2 ; a weight W is suspended from the extremity of the shorter arm. What force acting at the other end at an angle of 30° with the lever will support it?

10. A straight uniform lever of 10 lbs. weight and length 5 feet has its fulcrum at one end ; weights of 3 lbs. and 6 lbs. are attached to it at distances of 1 foot and 3 feet from the fulcrum, and it is kept horizontal by a vertical force at the other end. Find the pressure on the fulcrum.

11. Forces of 13 and 14 act on a bent lever at an angle whose cosine is $-\frac{5}{13}$. Find the pressure on the fulcrum.

12. Parallel forces balance each other on a straight lever, and the pressure on the fulcrum is ten times the difference between the forces. Find the ratio of the arms.

13. A B C is a piece of uniform wire bent at B into two straight arms such that the angle A B C is 135° . It is suspended from B, and rests with A B horizontal. Find the ratio of A B : B C.

14. A O B is a bent lever whose arms O A, O B are equal and straight. When weights P and Q are suspended from A and B it is in equilibrium with O A horizontal. What weight must be suspended from B so that the lever should be in equilibrium with O B horizontal?

15. A uniform lever of weight W is acted on at the ends A and B by weights P and Q. Show that if C be the fulcrum $A C : C B :: 2 Q + W : 2 P + W$.

16. A uniform heavy rod having a 10 lb. weight at one end balances about a point 4 feet from that end, and if the 10 lb. weight is replaced by a 5 lb. weight the fulcrum has to be moved 1 foot. Find the length and weight of the rod.

17. A O B is a bent lever whose arms A O, O B are at right-angles; O is the fulcrum. It is in equilibrium when weights P and Q are suspended from A and B, O A being inclined at an angle of 60° to the horizon. What force at B will keep P in equilibrium with O A inclined at an angle of 30° to the horizon?

Roman Steelyard.

18. A B is a Roman steelyard 10 feet long; its centre of gravity is 9 ins. from A, and the fulcrum 6 ins. from A. If the weight of the steelyard be 8 lbs., and that of the movable weight 1 lb., how far from B will the graduation 12 be? What is the greatest weight that can be weighed?

19. If a common steelyard lose one-tenth of its weight, the centre of gravity remaining unaltered, how would you correct the graduations?

20. In the common steelyard the distance of the centre of gravity from the fulcrum is 2 ins., and the movable weight 4 oz., and the weight of the beam 2 lbs. Where is the zero of graduation?

tion, and what is the least weight that can be weighed if the fulcrum is 3 ins. from the heavy end?

21. In a Roman steelyard the fulcrum is 8 ins. from the heavy end and 5 ins. from the centre of gravity, and lies between them; the weight of the steelyard is 4 lbs. and the movable weight 3 lbs. Where is the first graduation corresponding to 3 lbs.? What is the smallest weight that can be weighed? Where is the graduation for a weight of 15 lbs.?

22. A common steelyard is made of a uniform bar 18 ins. long and of 3 lbs. weight. It is suspended 3 ins. from one end, and the movable weight is 2 lbs. What is the greatest weight that can be weighed?

23. A common steelyard is formed of a uniform rod 1 foot long, the fulcrum being 1 inch from the end; the sliding weight and the weight of the rod are each 1 lb. If the sliding weight be changed into one of 2 lbs. find what error will be made by using the old graduations.

24. A uniform rod is divided into 20 equal parts, and the fulcrum is placed at the first graduation. The movable weight can move between the second division and the end, and substances from 2 lbs. up to 20 lbs. can be weighed. Show that the weight of the rod is $\frac{1}{5}$ lb. weight.

25. The beam of a steelyard is 33 ins. long, the body to be weighed being attached to the end A; the fulcrum is distant 4 ins. and the centre of gravity $5\frac{3}{4}$ ins. from A. The weight of the beam is 1 lb. weight, and the greatest weight that can be weighed 24 lbs. weight. Find the movable weight.

Danish Steelyard.

26. The distance between the zero of graduations and the end of the steelyard is divided into 20 equal parts, and the greatest weight that can be weighed is 3 lbs. 9 oz. Find the weight of the steelyard, and the weight which can be weighed when the fulcrum is at the first graduation.

27. A weight of 4 oz. is in equilibrium on a Danish steelyard when the fulcrum is 6 ins. from the end to which the weight is attached, and a weight of 8 oz. is in equilibrium when the fulcrum is 4 ins. from that end. Find the centre of gravity and the weight of steelyard.

28. The weight of a Danish steelyard is 1 lb., and the nearest distance of the fulcrum from the end to which the weight is hung is 1 inch, and the centre of gravity is 3 feet from that end. What is the greatest weight that can be weighed?

29. If the bar rests with the fulcrum half-way between the first and second graduations, show that the weight to be weighed is seven-fifths of the weight of the bar.

30. Find the length from the end of a Danish steelyard (1 lb. weight) to the zero of graduation when the distance between the graduations for 4 lbs. weight and 5 lbs. weight is 1 inch.

Balance.

31. The arms of a balance are $8\frac{3}{4}$ and 9 ins. respectively. What is the real weight of a substance which, when suspended from the longer arm, apparently weighs 27 lbs.?

32. A tradesman's balance has arms whose lengths are 11 and 12 ins., and it rests horizontally when the scales are empty. If he sell to each of two customers 1 lb. of tea at 2s. 9d. per lb., putting his weights into different scale-pans for the two transactions, how much does he gain or lose owing to his balance being incorrect?

33. The beam of a false balance is uniform and heavy; show that the arms are proportional to the differences between the true and the apparent weights.

34. A false balance has its beam horizontal when no weights are in the pans, but one arm is longer than the other by one-ninth. The seller puts the substance to be weighed as often into one scale as the other; show that he loses $\frac{5}{9}$ per cent. on his transactions.

35. The arms of a balance are 10 ins. and 11 ins. respectively. A body apparently weighs 10 lbs. when weighed from the shorter arm; what will it appear to weigh when weighed from the other arm?

36. If one of the pans of a common balance be loaded, and the apparent weights of a body are 18 oz. and 20 oz., find the amount of the loading.

37. The arms of a balance are unequal. What is the true weight of a body which apparently weighs 4 lbs. and 3 lbs. 1 oz.? Find also the ratio of the arms.

Wheel and Axle.

38. If the radius of the axle is 4 ins., and that of the wheel 3 feet, find what power will support a weight of 150 lbs.

39. Four sailors, each capable of raising 1 cwt., can just raise an anchor by means of a capstan whose radius is 1 ft. 2 ins., and whose spokes are 7 feet long (measured from the axis). Find the weight of the anchor.

40. The radii of a wheel and axle are 2 ft. and 2 ins. respectively, and the strings which hang from them support the two ends of a rod 2 ft. 2 ins. long and 10 lbs. weight. What weight must also hang from one of the strings that the rod may rest in a horizontal position?

Pulleys.

41. In the case of a single movable pulley whose weight is $\frac{1}{2}$ lb., and in which the two portions of the string are at right angles to each other, find the power required to support a weight of 10 lbs.

42. If the inclination of one branch of the string to the vertical be 30° , find the power necessary to support a weight W .

First System of Pulleys.

43. If the weight be 2 cwt. and the power the weight of 1 st., the weights of the pulleys being neglected, how many pulleys are there?

44. With three movable pulleys whose weights, beginning with the lowest, are 5 lbs., 4 lbs., and 3 lbs., and the weight 7 lbs., find the power.

45. If there be four movable pulleys, the weight of each being w , what power is required to support the pulleys only without any weight being attached?

46. In a system of three pulleys, each of which is 8 oz. weight, what power will support a weight of 5 lbs. attached to the lowest?

47. What weight can be supported by a power of 6 lbs. weight in a system of four pulleys each of 1 lb. weight?

48. With three movable pulleys whose weights, beginning with the lowest, are 4 lbs., 2 lbs., 1 lb. respectively, find what power will support a weight of 12 lbs.

49. In a system of four movable pulleys, each of which weighs 1 oz., find the weight which will be supported by a power of 1 lb. weight.

50. In a system of three movable pulleys, of weights 4 lbs., 5 lbs., and 6 lbs., find what weight a power of 1 cwt. will support.

51. Find the power required to support a 10 lb. weight when there are four movable pulleys each of 1 oz. weight.

52. What power will support a 54 lb. weight in a system of four movable pulleys whose masses, beginning from the lowest, are 4, 3, 2, 1 lbs.?

53. Show that if P and W be in equilibrium, and one of them be raised or lowered, the position of the centre of gravity remains the same.

54. By means of a system of three movable pulleys, each of 2 lbs. weight, a man of 12 st. weight raises himself in a basket attached to the lowest pulley by the exertion of a force of 22 lbs. weight. Find the weight of the basket.

55. How many movable pulleys, each weighing 1 lb., must be used in order that a power of 2 lbs. weight may support a weight of 65 lbs.? If the weight be raised 3 ins. how many feet of string must pass round the highest pulley?

Second System of Pulleys.

56. The two blocks each contain two pulleys, and the movable block weighs 8 lbs., what weight will 1 cwt. support?

57. There are nine pulleys in the two blocks, the lower of which weighs 8 lbs., what force will support a weight of 100 lbs.?

58. With eight pulleys, the weight of the lower block being 1 lb., what weight can a force of 2 lbs. weight raise?

59. Draw the arrangement of pulleys in this system so that five men, each capable of exerting a force equal to the weight of 1 cwt., may be able to raise $1\frac{1}{4}$ ton.

60. In this system of pulleys it is found that forces of 5 lbs weight and 6 lbs. weight can just support weights of 18 lbs. and 22 lbs. attached to the lower block. Find the number of strings and the weight of the lower block.

Third System of Pulleys.

61. If the weight be 2 cwt. 1 lb., and the power 1 st. 1 lb., the weights of the pulleys being neglected, how many pulleys are there?

62. With two movable pulleys of weights 1 lb. and 2 lbs. what weight could be supported by a power of 3 lbs.?

63. With two pulleys, of which the lower weighs 5 lbs. weight, what weight will a power of 3 lbs. weight support?

64. If there be three pulleys, each weighing 4 oz., find the power necessary to support a weight of 8 lbs.

65. In a system of four pulleys, each weighing 2 oz., find the weight which will be supported by a power of 1 lb. weight.

66. In a system of four movable pulleys, the weight of each being w , what weight will be supported without the application of any power?

67. Find the power required to support a weight of 11 lbs. when there are four pulleys (three movable) each of 1 oz. weight.

68. If there be four movable pulleys, each of 1 lb. weight, and the weight attached to the lowest be 3 cwt., find the power.

69. With three movable pulleys, of weights w_1 , w_2 , w_3 , a force P balances W ; if the first and third pulleys be interchanged P_1 balances W . Show that $P_1 - P = \frac{2}{3}(w_1 - w_3)$.

70. If there be three movable pulleys whose masses, beginning with the lowest, are 1, 2 and 3 lbs., what power is required to support a weight of 91 lbs.?

71. If this system be modified by making the string which passes over each pulley pass under a small pulley attached to the weight and then attaching it to the former pulley, show that for n pulleys, whose weights may be neglected, the mechanical advantage will be $3^n - 1$.

Inclined Plane.

72. A weight of 12 lbs. is supported on a plane of an angle of 30° by a force acting up the plane. Find this force.

73. What force acting horizontally will support a weight of 12 lbs. on a plane whose angle is 60° ?

74. What is the inclination of the plane on which a power parallel to the plane will support twice its own weight?

75. Find the inclination of the plane on which a horizontal force will support a weight equal to itself.

76. What power acting up the plane will support a weight of 52 lbs. on a plane whose height is to its base as 5 is to 12?

77. A particle weighing 9 lbs. is kept at rest by a horizontal force of $3\sqrt{3}$ lbs. weight. Find the angle of the plane and the pressure upon it.

78. If the weight, power, and pressure on the plane are in the ratio $\sqrt{3} : 1 : 1$, find the inclination of the plane and the direction of the force.

79. A weight W is supported on a plane whose angle is 45° by a force P such that $W = \sqrt{\frac{3}{2}}$. Find the direction of P .

80. The angle of the plane is $\cos^{-1} \frac{1}{3}$, and there is equilibrium when the force acts up the plane. If the same force now act horizontally, show that an equal force applied in the proper direction will still preserve equilibrium, and find the direction of this force.

81. The angle of a plane is 30° , and a force P , acting horizontally, keeps the weight in equilibrium. If P now act at an angle of 30° with the plane, show that there will still be equilibrium, and that the pressure on the plane will be reduced one-half.

82. A weight W is supported on a plane whose angle is 30° by a force P such that $W = P\sqrt{2}$. Find the direction of P .

83. Find the direction in which the power must act in order that the pressure on the plane may be double the pressure when the power is the least possible.

84. Two planes, whose angles are 60° and 45° respectively, have the same altitude and are placed back to back; two weights connected by a string passing over their common vertex are in equilibrium upon them. Compare these weights.

85. A rope inclined to the vertical at an angle of 30° is just strong enough to support 180 lbs. weight on a plane whose angle is 30° . Find approximately the greatest weight it would support if hanging vertically.

86. A weight $2P$ is kept in equilibrium on a plane by a horizontal force P and an equal force acting up the plane. Find the ratio of the length, base, and height of the plane, and the pressure of the plane.

87. A weight W is kept in equilibrium by two forces P acting parallel to the plane and horizontally. The angle of the plane is α , such that $\sin \alpha = \frac{5}{13}$; show that $W = 5 P$.

88.* If the pressure on the plane be an arithmetic mean between the weight and the power, and the inclination of the power to the horizon be double that of the plane, prove that the sine of the angle of inclination of the power to the horizon will be $\frac{3}{4}$.

89. W is supported on an inclined plane (1) by a power acting up the plane, (2) by a power acting horizontally. Show that $W = \sqrt{R R_1}$, where R, R_1 are the pressures on the plane in the two cases.

90. If P be the power required to support a weight W on an inclined plane, and the plane be then placed so that its vertical side is now the base and P_1 is the power required, show that $W = \sqrt{P^2 + P_1^2}$.

91. A power of 3 lbs. weight can support 5 lbs. weight on an inclined plane; the plane is now turned on to the other side as base. Show that the same power can only support a weight of $3\frac{3}{4}$ lbs.

92. Find the direction in which a force of 8 lbs. weight must act in order that it should just support a weight of 12 lbs. on a plane whose angle is 30° .

93. If the inclined plane is the upper surface of a smooth wedge whose under surface rests on a smooth horizontal table, find the horizontal force necessary to keep the wedge from sliding on the table.

94. A weight of 7 cwt. is supported on a smooth plane inclined at an angle of 30° to the horizon by means of a rope which, after passing over a smooth pulley at the top of the plane, is wound round the axle of a windlass below, the handle describing a circle of 16 ins. radius and the radius of the axle being 4 ins. Find the pressure on the handle.

Screw.

95. A screw having its threads $\frac{1}{2}$ inch apart is turned by an arm 7 ins. long, and moves in a smooth nut with its axis vertical. What force must be applied at the end of the arm to support a body of 11 st.?

CHAPTER IX

FRICTION

129. IN the preceding chapter we have supposed that when two surfaces are in contact the mutual reaction between them is always along their common normal ; in other words, we have supposed that the surfaces of the bodies in contact were perfectly smooth. In nature this is only true when the surfaces in contact have no tendency to move the one against the other, as in the case of a body resting on a horizontal table. In this case the weight of the body acting vertically downwards can only be supported by the pressure of the table, which must therefore act vertically upwards—that is, perpendicular to the surface of the table.

Now suppose the table to be tilted, so that it is no longer horizontal ; the resultant pressure of the table must still act vertically upwards in order to support the weight, and consequently cannot be perpendicular to the table. It may in fact be resolved into two forces R and F , of which R acts normal to the surface, and F acts along the surface up the line of greatest slope, and consequently in a direction opposite to that in which the body would move if it could.

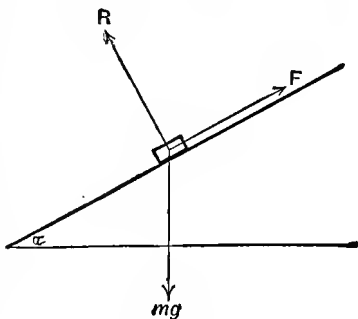


FIG. 95.

In this case R is called the *normal pressure*, and F is called the *force of friction*.

This force of friction is due to the *roughness* of the two

Note.—In fig. 95, $mg = W$ (Art. 80).

surfaces, and is always called into action whenever a body has a tendency to move upon the surface with which it is in contact ; but only so much is called into action as is required to prevent motion. Hence we say that friction is a *passive* force and not an active force—it prevents motion, but never causes it.

The amount of friction that can be called into action in any particular case is necessarily limited.

Thus, in the case of the body on the slanting table, as we increase the angle of inclination, the body has a greater tendency to slip, and the force of friction increases so as always to prevent motion up to a certain point ; after which, if we increase the angle of inclination, F is no longer able to resist the tendency to motion, and the body begins to slide.

The amount of friction which is called into action when the body is just on the point of sliding is called the *limiting friction*.

If we resolve these forces R , F , W perpendicular to and along the plane, we must have

$$R = W \cos \alpha,$$

$$F = W \sin \alpha,$$

and

$$W = \sqrt{R^2 + F^2},$$

α being the angle of inclination of the plane to the horizon

Hence so long as there is equilibrium $F = W \sin \alpha$; thus F increases as α increases, until the *limiting position of equilibrium* is reached, when F can increase no longer. If α is still increased, the body will begin to move with an acceleration

$$\frac{W \sin \alpha - F}{m}, \text{ } m \text{ being the mass of the body. (See Art. 27.)}$$

The amount of *limiting friction* which can be called into action between two substances varies with the nature of the substances ; some materials, being much rougher than others, are consequently able to exert a much greater limiting friction. In nature no substances are so smooth as to be incapable of exerting a small resistance to motion, though it is possible so to polish the surfaces in contact as to make this resistance very small indeed.

In speaking of a *smooth* surface we do not merely mean a surface which is not very rough, but a surface which is *perfectly smooth*—that is, which offers no resistance to motion. We shall consider

such surfaces in future examples for the sake of illustration, though, like much of the work in the earlier stages of a physical subject, they must be regarded as purely hypothetical.

130. We will now proceed to describe briefly how the laws of limiting friction were experimentally determined by Coulomb in the last century.

Take a number of blocks of the *same* material but of various shapes (not spherical), and place them upon a table with a uniform surface; now gradually raise one end of the table; it will be found that when the limiting position of equilibrium is reached for one block, it is reached for all—that is to say, they will all begin sliding together. If we now change the materials, we find that blocks of *different* material begin to slide at different inclinations of the table.

We therefore infer that the limiting friction does not depend upon the size or the shape of the surfaces in contact, provided that one cannot roll on the other. Further, since the ratio $\frac{F}{R}$ or $\frac{F}{W}$ depends only on α in the preceding example, and since α remains the same for the same substances, we infer that the limiting friction varies as the normal pressure.

131. We may now state the **Laws of Statical Friction** :

Law I.—The direction of the force of friction is opposite to the direction in which motion would take place if there were no friction.

Law II.—In all cases of equilibrium the magnitude of the force of friction is equal to the least force required to prevent motion.

Laws of Limiting Friction.

Law III.—So long as the substances in contact remain the same the limiting friction varies as the normal pressure.

Law IV.—So long as the normal pressure remains the same the limiting friction is independent of the size and shape of the surfaces in contact.

Definition.—The constant ratio of the limiting friction to the normal pressure is called the **coefficient of friction**.

Thus, if μ = coefficient of friction,

$$\frac{F}{R} = \mu; \quad \therefore F = \mu R.$$

Coulomb also discovered that Laws III. and IV. of statical friction were also true when the body was in motion, but that in this case the friction exerted opposite to the direction of motion was less than the limiting statical friction. Hence the coefficient of dynamical friction is less than the coefficient of statical friction.

He also found that the dynamical friction was independent of the velocity of the moving body. Though this is not rigidly accurate, it is sufficiently so to be embodied in our law of dynamical friction.

Law V.—The coefficient of dynamical friction is less than the coefficient of statical friction, but is independent of the velocity with which the body is moving.

The value of μ varies greatly for different substances, being much greater for substances like wood and stone than for metal, glass, or marble. It can usually be considerably lessened by smearing the surfaces in contact with oil or soap.

Rolling friction is much less than sliding friction.

132. The actual pressure between two substances is the resultant of the normal pressure and the force of friction.

Definition.—The angle between the resultant pressure in the limiting position of equilibrium and the normal is called the **angle of friction**.

Thus, if ϕ be the angle of friction, OA the direction of the resultant of the normal pressure R and the limiting friction F , represented by OB , OC respectively, then, completing the rectangular parallelogram $BOCA$, we have

$$\tan \phi = \frac{BA}{BO} = \frac{F}{R} = \frac{\mu R}{R} = \mu.$$

$$\text{Hence } \phi = \tan^{-1} \mu.$$

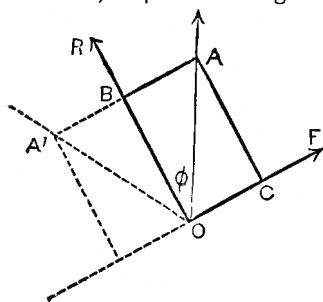


FIG. 56.

Since the resultant reaction cannot be inclined to the normal at a greater angle than ϕ , it cannot act outside the angle $\angle AOA'$ when acting in the plane of the paper.

Hence it cannot act outside the cone whose axis is the normal OB and vertical angle 2ϕ . This cone is called the **cone of friction**.

133. To find the coefficient of friction between any two substances experimentally.

We take a plane board of the one substance hinged along one edge, so that it can revolve about that edge; on it we place a portion of the other substance with a plane base, and turn the first board about its fixed edge until the other substance is in the limiting position of equilibrium.

Thus, let W be the weight of the substance, R the normal pressure, μR the limiting friction, so that μ is the coefficient of friction between the two substances.

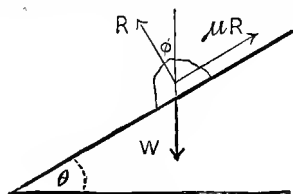


FIG. 97.

Let θ be the angle of inclination of the plane to the horizon in the limiting position of equilibrium. Resolving the forces parallel and perpendicular to the plane we have

$$\begin{aligned} R &= W \cos \theta, \\ \mu R &= W \sin \theta; \\ \therefore \frac{\mu R}{R} &= \frac{W \sin \theta}{W \cos \theta}; \\ \therefore \mu &= \tan \theta. \end{aligned}$$

θ is carefully measured, and thus μ may be determined.

Note.—Since $\mu = \tan \phi$, where ϕ is the angle of friction, it follows that $\theta = \phi$.

Hence the angle of inclination of the plane in the limiting position of equilibrium is equal to the angle of friction. For this reason the angle $\tan^{-1}\mu$ is often called the *angle of repose*.

This result might also have been obtained from the following considerations :—

ϕ is the angle between the resultant pressure and the normal pressure—that is, between the direction of R and the vertical ;

θ is the angle between the plane and the horizontal ; the lines which contain the angle ϕ are respectively perpendicular to the lines which contain the angle θ ;

$$\therefore \theta = \phi.$$

The following table shows the value of μ in various cases, and is taken from Morin's 'Notions Fondamentales de Mécanique' :—

Surfaces.						μ
Wood on wood, dry	.	.	.	.	.	0·25 to 0·5
do. soaped	.	.	.	.	.	0·2
Metals on oak, dry	.	.	.	.	.	0·5 to 0·6
do. wet	.	.	.	.	.	0·24 to 0·26
do. soaped	.	.	.	.	.	0·2
Metals on elm, dry	.	.	.	.	.	0·2 to 0·25
Hemp on oak, dry	.	.	.	.	.	0·53
do. wet	.	.	.	.	.	0·33
Leather on oak, wet or dry	.	.	.	.	.	0·27 to 0·35
Leather on metals, dry	.	.	.	.	.	0·56
do. wet	.	.	.	.	.	0·36
do. greasy	.	.	.	.	.	0·23
do. oiled	.	.	.	.	.	0·15
Metals on metals, dry	.	.	.	.	.	0·15 to 0·2
do. wet	.	.	.	.	.	0·3
Smooth surfaces with unguents, occasionally greased	.	.	.	.	.	0·07 to 0·08
do. do. well greased	.	.	.	.	.	0·05
do. do. do. best results	.	.	.	.	.	0·03 to 0·036

('Encyclopædia Britannica,' 9th edition, vol. xv., page 765.)

We will now proceed to discuss the conditions of equilibrium in the case of the inclined plane, the screw, and the wedge when they are not perfectly smooth.

The action of rough surfaces on strings is of too complicated a nature to be considered in this book, so that we shall not now be able to discuss the equilibrium of the wheel and axle, and the various systems of pulleys when they are rough.

134. *Condition of equilibrium on a rough inclined plane when the power acts parallel to the plane.*

Let μ = the coefficient of friction, and let us suppose that the weight W is on the point of moving *up* the plane under the action of the power P , so that the force of friction acts *down* the plane.

Resolving the forces parallel and perpendicular to the plane we have :

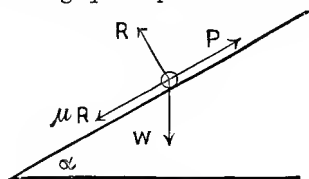


FIG. 98.

$$P = \mu R + W \sin \alpha,$$

$$R = W \cos \alpha ;$$

$$\begin{aligned} \therefore P &= \mu W \cos \alpha + W \sin \alpha \\ &= W (\sin \alpha + \mu \cos \alpha). \end{aligned}$$

Similarly, if the body were on the point of moving *down* the plane,

$$P = W (\sin \alpha - \mu \cos \alpha).$$

It is obvious in this case that α , the angle of the plane, must be greater than the *angle of friction*, since if it were not the weight would rest on the plane without any power being applied.

Hence, if the angle of the plane be greater than the angle of friction, the force required to maintain equilibrium must lie between $W (\sin \alpha \pm \mu \cos \alpha)$.

134A. *Rough inclined plane when the power makes an angle θ with the plane.*

Suppose W to be on the point of moving up the plane ; then, with the usual notation, and resolving the forces along and perpendicular to the plane, we have

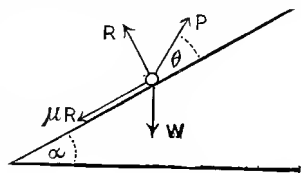


FIG. 99.

$$P \cos \theta = W \sin \alpha + \mu R,$$

$$P \sin \theta = W \cos \alpha - R.$$

Eliminate R ,

$$P \cos \theta = W \sin \alpha + \mu (W \cos \alpha - P \sin \theta),$$

$$P (\cos \theta + \mu \sin \theta) = W (\sin \alpha + \mu \cos \alpha),$$

$$P = W \cdot \frac{\sin \alpha + \mu \cos \alpha}{\cos \theta + \mu \sin \theta}.$$

So also if the body were on the point of moving down the plane, μR would act up the plane, and we should have

$$P = W \cdot \frac{\sin \alpha - \mu \cos \alpha}{\cos \theta - \mu \sin \theta}.$$

134B.* To find the magnitude and direction of the least force which will move a body up a rough inclined plane.

Using the first result of the previous article, we have, when W is on the point of moving up the plane,

$$P = W \cdot \frac{\sin \alpha + \mu \cos \alpha}{\cos \theta + \mu \sin \theta}.$$

If ϕ be the angle of friction, $\mu = \tan \phi$,

$$\begin{aligned} P &= W \cdot \frac{\sin \alpha + \mu \cos \alpha}{\cos \theta + \mu \sin \theta} \\ &= W \cdot \frac{\sin \alpha + \tan \phi \cdot \cos \alpha}{\cos \theta + \tan \phi \sin \theta} \\ &= W \cdot \frac{\sin \alpha \cos \phi + \cos \alpha \sin \phi}{\cos \theta \cos \phi + \sin \theta \sin \phi} \\ &= W \cdot \frac{\sin (\alpha + \phi)}{\cos (\theta - \phi)}. \end{aligned}$$

P is a minimum when $\cos (\theta - \phi)$ is a maximum ;

i.e. when $\cos (\theta - \phi) = 1$,

when $\theta - \phi = 0$,

when $\theta = \phi$,

and then

$$P = W \sin (\alpha + \phi).$$

Hence the least force which will just move a weight W up a rough inclined plane must act at an angle with the plane equal to the angle of friction (ϕ), and be just greater than

$$W \sin (\alpha + \phi).$$

¹ This result might have been obtained at once by resolving P and W perpendicular to the direction of the total resistance, thus $P \cos (\theta \pm \phi) = W \sin (\alpha \mp \phi)$.

Similarly, if W is on the point of moving down the plane,

$$P = W \cdot \frac{\sin(a - \phi)}{\cos(\theta + \phi)}.$$

In this case P is least when $\theta = -\phi$.

Hence the least force which will just keep a weight W from moving down a rough inclined plane must act below the plane at an angle equal to the angle of friction, and is equal to $W \sin(a - \phi)$.

Note.—These two values of P are the same as the powers required to keep W at rest on two *smooth* inclined planes whose inclinations to the horizon are $a + \phi$ and $a - \phi$ respectively. In the first case the effect of the limiting friction is the same as if the angle of the plane were increased by an angle equal to the angle of friction; and in the second case the effect is the same as if the angle of the plane were diminished by an angle equal to the angle of friction.

135. *Equilibrium on a rough screw.*

With the notation used for the smooth screw in Article 126, suppose that the screw is on the point of rising, so that the friction μr acts down the tangent at the point O :

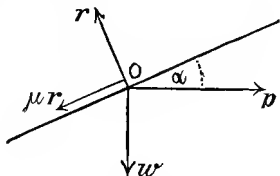


FIG. 100.

$$\cos a = w \sin a + \mu r,$$

$$r = w \cos a + \mu \sin a;$$

$$\therefore p \cos a = w \sin a + \mu (w \cos a + p \sin a),$$

$$p (\cos a - \mu \sin a) = w (\sin a + \mu \cos a),$$

$$p = w \cdot \frac{\sin a + \mu \cos a}{\cos a - \mu \sin a}.$$

$$\therefore \Sigma(p) = \Sigma \left(w \cdot \frac{\sin a + \mu \cos a}{\cos a - \mu \sin a} \right)$$

$$= W \cdot \frac{\sin a + \mu \cos a}{\cos a - \mu \sin a}.$$

¹ See note on previous page.

² This result might have been obtained from the result of Art. 134A by putting $\theta = a$.

Again, $P \cdot b = \Sigma (p a) = a \Sigma (p)$

$$= a \cdot W \cdot \frac{\sin a + \mu \cos a}{\cos a - \mu \sin a};$$

$$\therefore \frac{P}{W} = \frac{a}{b} \cdot \frac{\sin a + \mu \cos a}{\cos a - \mu \sin a}.$$

* This result may be written in another form by substituting $\mu = \tan \phi$:

$$\begin{aligned} \frac{P}{W} &= \frac{a}{b} \cdot \frac{\sin a + \tan \phi \cos a}{\cos a - \tan \phi \sin a} \\ &= \frac{a}{b} \cdot \frac{\sin (a + \phi)}{\cos (a + \phi)} \\ &= \frac{a}{b} \cdot \tan (a + \phi). \end{aligned}$$

which is the same result that we should have obtained with a *smooth* screw whose *pitch* was $a + \phi$.

Similarly, if the screw were on the point of moving down, we should get

$$\frac{P}{W} = \frac{a}{b} \cdot \tan (a - \phi).$$

Rough Wedge.

136. Using the description and notation employed in Article 127, and supposing that the wedge is on the point of moving down, we have

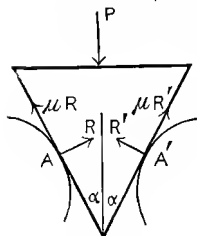


FIG. 101.

$$\begin{aligned} P &= (R + R') \sin a + \mu (R + R') \cos a. \\ R \cos a - \mu R \sin a &= R' \cos a - \mu R' \sin a; \\ \therefore R &= R'. \end{aligned}$$

$$\begin{aligned} \text{Hence, } P &= 2 R \sin a + 2 \mu R \cos a \\ &= 2 R (\sin a + \mu \cos a). \end{aligned}$$

Similarly, if the wedge were on the point of moving upwards,

$$P = 2 R (\sin a - \mu \cos a).$$

It is obvious that in this case it is possible for the wedge to be in equilibrium under the action of the pressures alone without any force being applied at the top.

Thus, putting $P = 0$, we have

$$\sin \alpha - \mu \cos \alpha = 0 ; \therefore \tan \alpha = \mu.$$

Hence the wedge will be in equilibrium without any force being applied if its vertical angle is equal to or less than twice the angle of friction.

Note.—In this case the resultant pressures at A and A' must be horizontal.

Example 1.—A uniform ladder rests against a vertical wall with its lower end on a horizontal pavement. If the coefficients of friction between the ladder and the pavement and between the ladder and the wall be μ and μ' respectively, determine the greatest inclination the ladder can make with the vertical.

Let AB be the ladder, $2l$ its length, θ the angle it makes with the vertical, R and R' the normal pressures at A and B respectively, μR , $\mu' R'$ the limiting friction, W the weight of the ladder acting through C the middle point.

Resolving the forces vertically and horizontally,

$$R + \mu' R' = W \quad \dots (1)$$

$$R' = \mu R \quad \dots (2)$$

Take moments about B :

$$R \cdot 2l \sin \theta - \mu R \cdot 2l \cos \theta - W \cdot l \sin \theta = 0 \quad \dots (3)$$

$$\text{From (1) and (2)} \quad R(1 + \mu\mu') = W.$$

Substitute for W in (3)

$$R \cdot 2 \sin \theta - \mu R \cdot 2 \cos \theta - R(1 + \mu\mu') \sin \theta = 0,$$

$$\sin \theta (1 - \mu\mu') = 2\mu \cos \theta,$$

$$\tan \theta = \frac{2\mu}{1 - \mu\mu'}.$$

Note.—The resultant pressures at A and B and the direction of the weight must all pass through one point O.

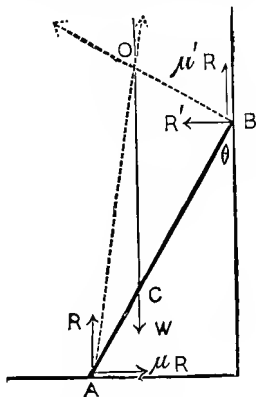


FIG. 102.

Corollary.—If the wall be smooth $\mu' = 0$, then $\tan \theta = 2\mu$.
(See Example 2 at the end of Chapter vii.).

If the ground is smooth $\mu = 0$; then $\tan \theta = 0$,

Hence equilibrium is impossible. This is also obvious by Art. 96.

* If there is a man of weight w on the ladder, let D be his position, and let the ratio $\frac{BD}{BA} = r$, so that our equations must be modified as follows :

$$\begin{aligned} R + \mu' R' &= W + w, \\ R' &= \mu R, \end{aligned}$$

$$R \cdot 2l \sin \theta' - \mu R \cdot 2l \cos \theta' - W \cdot l \sin \theta' - w \cdot 2lr \sin \theta' = 0;$$

since $BD = 2lr$.

Here
$$R = \frac{W + w}{1 + \mu\mu'};$$

$$\therefore \frac{W + w}{1 + \mu\mu'} \cdot 2 \sin \theta' - W \cdot \sin \theta' - 2rw \sin \theta' = \frac{\mu(W + w)}{1 + \mu\mu'} \cdot 2 \cos \theta';$$

$$\begin{aligned} \tan \theta' &= \frac{2\mu(W + w)}{2(W + w) - (W + 2rw)(1 + \mu\mu')} \\ &= \frac{2\mu(W + w)}{W(1 - \mu\mu') + 2w - 2wr(1 + \mu\mu')} \\ &= \frac{\frac{2\mu}{1 - \mu\mu'} \left(1 + \frac{w}{W}\right)}{1 + \frac{2w}{W} \left(\frac{1 - r(1 + \mu\mu')}{1 - \mu\mu'}\right)}. \end{aligned}$$

Now $\frac{2\mu}{1 - \mu\mu'} = \tan \theta$, where θ is the inclination without the man;

$$\therefore \tan \theta' = \frac{1 + \frac{2w}{W} \cdot \frac{1 - r(1 + \mu\mu')}{1 - \mu\mu'}}{1 + \frac{2w}{W}}.$$

Hence $\tan \theta$ is $>$ or $<$ $\tan \theta'$,

according as $2 \cdot \frac{1 - r(1 + \mu\mu')}{1 - \mu\mu'} >$ or $<$ 1 ;

$$\text{i.e. as } 2 - 2r(1 + \mu\mu') > \text{ or } < 1 - \mu\mu',$$

$$\text{as } 2r(1 + \mu\mu') < \text{ or } > 1 + \mu\mu',$$

$$\text{as } 2r \text{ is } < \text{ or } > 1,$$

$$\text{as } r < \text{ or } > \frac{1}{2}.$$

Hence, θ is $<$ θ' if r is $>$ $\frac{1}{2}$; that is, if the man is on the lower half of the ladder; so that if the ladder is on the point of slipping

down when a man gets on it, it will be less inclined to slip so long as he is on the lower half of it, but as soon as he gets beyond the middle it will slip. (Encyc. Brit. 'Mechanics.')

Example 2.—A brick is placed on its end on a rough plane and the plane is tilted. Find the proportion of its dimensions in order that it should slide before it tumbles over.

Let $A B C D$ be the brick standing on the base $A B$, and let a, b be the length and depth of the brick respectively; its breadth perpendicular to the plane of the paper will not affect the question. Let G be the

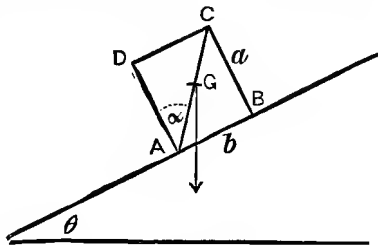


FIG. 103.

centre of gravity of the brick, and let $C A D = a$.

Let θ be the inclination of the plane to the horizon; then θ will also be the inclination of $D A$ to the vertical. The brick will be on the point of sliding when θ is equal to ϕ the angle of friction.

The brick will be on the point of falling over when the vertical through the centre of gravity passes through A the extremity of the base; that is, when $\theta = a$.

When $\theta > \phi$ the brick will slide;

when $\theta > a$ the brick will fall over;

$\therefore$ if $\phi < a$ the brick will slide before it falls;

$\tan \phi$ must be less than $\tan a$,

$$\mu < \frac{b}{a};$$

$$\therefore b > \mu a.$$

This gives the ratio of the depth of the brick to its length.

Example 3.—A weight W is attached by a string to a point O on a rough inclined plane whose inclination α is greater than the angle of friction, so that the body can only describe the arc of a circle whose centre is O . What is the greatest angle that the string can make with the line of greatest slope?

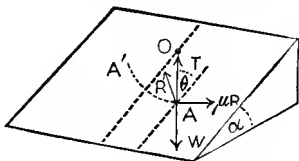


FIG. 104.

Let θ be the angle the string makes with the line of greatest slope in the limiting position of equilibrium; AA' the arc of the circle the body can describe.

Then the forces which act on the body are

$$\begin{array}{l} W \text{ vertically downwards,} \\ R \perp^r \text{ to the plane,} \\ T \text{ along the string } AO, \\ \mu R \text{ along the tangent to the circle at } A \end{array} \left. \vphantom{\begin{array}{l} W \text{ vertically downwards,} \\ R \perp^r \text{ to the plane,} \\ T \text{ along the string } AO, \\ \mu R \text{ along the tangent to the circle at } A \end{array}} \right\} \begin{array}{l} \text{These two are} \\ \text{in the plane.} \end{array}$$

W may be resolved into $W \cos \alpha \perp^r$ to the plane, and $W \sin \alpha$ along the line of greatest slope, of which $W \cos \alpha$ is in equilibrium with R , and $W \sin \alpha$ is in equilibrium with T and μR ;

$$\therefore R = W \cos \alpha \quad \dots \dots \dots (1)$$

Now, resolving $W \sin \alpha$ parallel and perpendicular to AO in the plane, we have

$$T = W \sin \alpha \cdot \cos \theta \quad \dots \dots \dots (2)$$

$$\mu R = W \sin \alpha \cdot \sin \theta \quad \dots \dots \dots (3)$$

Substituting for R from (1) in (3)

$$\mu W \cdot \cos \alpha = W \sin \alpha \cdot \sin \theta ;$$

$$\therefore \sin \theta = \mu \cot \alpha.$$

This gives the greatest value θ can have. T can then be found by substituting for θ in equation (2).

Example 4.—A rough vertical screw is in equilibrium without any power being applied. Find the least number of revolutions the thread can be allowed to make in a given length l of the screw.

There will be the least number of revolutions of the thread when the pitch of the screw is greatest. In the limiting position of equilibrium, when the screw is on the point of moving downwards, the pitch of the screw is equal to the angle of friction.

Hence, if a be the greatest pitch of the screw,

$$\tan \alpha = \mu.$$

Now if a be the radius of the cylinder of the screw,

$2\pi a \cdot \tan \alpha =$ vertical distance between two consecutive revolutions ;

$\therefore 2\pi a \cdot \mu =$ greatest distance between two revolutions.

Hence, if n be the least number of revolutions in a given length l of the screw,

$$n \cdot 2\pi a \mu = l ;$$

$$\therefore n = \frac{l}{2\pi \mu a}.$$

Example 5.—A picture is hung from a peg in a rough wall by a string attached to the peg and the middle point of the upper side of the picture, so that the length of the string (l) is equal to the height of the picture. Determine the angle θ which the picture makes with the wall in the limiting position of equilibrium, and the tension of the string when the picture is in that position.

Let AB represent the cross section of the picture, W its weight acting through C , $O A$ the string, so that $O A$, $A B$ make equal angles with the wall.

Let R be the normal pressure at B , μR the limiting friction acting up the wall ;

T the tension of the string.

Then, resolving the forces vertically and horizontally,

$$W = T \cos \theta + \mu R,$$

$$R = T \sin \theta ;$$

$$\therefore W = T (\cos \theta + \mu \sin \theta).$$

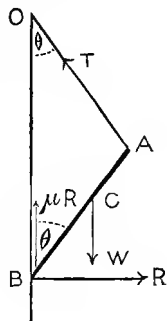


FIG. 105.

Take moments about B .

$$W \cdot BC \sin \theta = T \cdot BO \sin \theta,$$

$$W \cdot \frac{l}{2} = T \cdot BO,$$

$$T (\cos \theta + \mu \sin \theta) \frac{l}{2} = T \cdot 2l \cos \theta ;$$

$$\therefore \cos \theta + \mu \sin \theta = 4 \cos \theta ;$$

$$\therefore \mu \sin \theta = 3 \cos \theta,$$

$$\tan \theta = \frac{3}{\mu}.$$

$$\text{Hence } \sin \theta = \frac{3}{\sqrt{9 + \mu^2}}, \text{ and } \cos \theta = \frac{\mu}{\sqrt{9 + \mu^2}} ;$$

$$\therefore W = T \left(\frac{\mu}{\sqrt{9 + \mu^2}} + \frac{3\mu}{\sqrt{9 + \mu^2}} \right)$$

$$= T \left(\frac{4\mu}{\sqrt{9 + \mu^2}} \right) ;$$

$$\therefore T = W \cdot \frac{\sqrt{9 + \mu^2}}{4}.$$

Examples.

1. Find the greatest and least forces required to support a 10 lb. weight on a rough plane inclined at an angle $\tan^{-1} \frac{3}{4}$ to the horizon, the coefficient of friction being $\frac{1}{4}$, the forces acting parallel to the plane.

2. A 5 lb. weight can just be supported on a rough inclined plane by a 2 lb. weight, or can just support a 4 lb. weight suspended by a string passing over a smooth pulley at the top of the plane. Find the coefficient of friction and the inclination of the plane to the horizon.

3. A straight uniform rod A B of weight W rests with its lower end A on a smooth horizontal plane, and its upper end B supported by a rough peg ($\mu = \frac{1}{\sqrt{3}}$). Show that the inclination of the rod to the horizon is 30° , and find the pressure on the plane at A.

4. A uniform heavy beam rests with one end on a rough horizontal plane and the other against a rough vertical wall, and is inclined at an angle of 30° to the horizon. Find μ , supposed to be the same for both surfaces.

5. A body of weight W can just rest on a rough inclined plane, inclined at an angle of 30° to the horizon. What is the least force acting (1) along the plane, (2) horizontally, which will make it begin to move up the plane? Determine the pressure in the two cases.

6. Find the tangent of the angle at which a heavy uniform beam will rest against a smooth vertical wall, with its lower end on a horizontal rough plane on which the resistance to sliding is two-thirds of the weight of the beam.

7. A weight rests on a rough inclined plane, and is just prevented from sliding down the plane by a given force P. Show that the weight supported will be the greatest when P is inclined to the plane at an angle equal to the angle of friction measured below the plane.

8. A heavy particle is in equilibrium on a rough plane inclined to the horizon at twice the angle of friction, being just supported by a force whose direction bisects the angle between the plane and the horizon. Show that the resolved part of the weight in the direction of the force just balances the force, and find the pressure on the plane.

9. A ladder 13 feet long rests with one end on a rough pavement and the other against the smooth vertical wall of a house at a height of 12 feet from the ground. A man ascends it. Show that the frictional resistance of the ground must increase as he ascends, and if the coefficient of friction be less than $\frac{1}{13}$, the ladder (whose weight may be neglected) will slip before he reaches the top.

10. A heavy uniform straight rod rests with its extremities on two fixed rough similar rods, each inclined at an angle of 45° to the horizon, all three being in the same vertical plane. Find the greatest angle at which the movable rod can be inclined to the horizon.

11. On a rough horizontal floor is placed a uniform equilateral triangle ABC with its base BC vertical. The vertex A is in contact with a smooth sloping wall, which is perpendicular to the plane of the triangle and makes an angle of 120° with the floor. If the triangle just rests in this position, find the coefficient of friction.

12. A weight is supported on a rough inclined plane by a force acting parallel to the plane, whose inclination is α and coefficient of friction μ . If the force which would just pull it up the plane is double the force which just prevents it from slipping down, prove that $\tan \alpha = 3\mu$.

13. A and B are two small equal heavy rings which slide on a rough horizontal rod, the coefficient of friction being $\frac{1}{3}$. Another equal heavy ring C slides on a smooth weightless string connecting A and B , and hangs below the rod in a position of limiting equilibrium. Show that A, B, C are the corners of an equilateral triangle.

14.* Two particles, whose weights are equal, are connected by a light string, and rest on a rough vertical circle, the coefficient of friction being $\tan \epsilon$. Show that when there is no friction between the string and the circle, and both weights are on the point of motion, the radius of the circle through the middle point of the string will make an angle ϵ with the vertical.

15.* A uniform rod rests at an angle α to the horizon, with one end on a rough horizontal plane, and on the point of slipping, and the other end supported by a string inclined at an angle β to the horizon. Prove that the coefficient of limiting friction is

$$\frac{1}{\tan \beta - 2 \tan \alpha}.$$

CHAPTER X*

VIRTUAL WORK

137. It has been stated at the end of Chapter VIII. that the use of a machine is to transfer *work*. Now, when is a force said to do *work*?

A force is said to do work when its point of application is displaced in the direction in which the force acts.

Thus, let A be the point of application of the force F ; suppose that, under the action of F , A is displaced to A' , and draw $A'a$ perpendicular to the line of action of F ; then Aa



FIG. 106

is the projection of the displacement of A on the line of action of F , and $F \cdot Aa$ is the work done by F during the displacement. If a falls on the side of A towards which

F acts, the force is said to do work, and the work is said to be *positive*; if a falls on the other side, work is said to be done *against* the force, and is *negative*.

Thus $F \cdot s$ is always the measure of the work done by or against F during any displacement; s being the projection of the displacement on the direction of the force, which will be positive or negative according to the magnitude of the angle which AA' makes with F .

Energy is the **capacity of doing work**. All forces have energy, and when a force does work some of its energy disappears; but it is not lost, it is merely transferred to some other part of the system.

This is the Principle of the Conservation of Energy, which may be stated as follows:

The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible. (Clerk Maxwell, 'Matter and Motion,' p. 60.)

This great principle is really embodied in Newton's Third Law of Motion—

Action and Reaction are equal and opposite.

No work can be done without the expenditure of an equal quantity of work. There can be no increase of energy in any system without the loss of at least an equal quantity of energy in some other system. If, in any conservative system, one form of energy is increased, it must be done at the expense of some other form of energy in the system.

The different forms of energy are various ; thus, mechanical energy may be transformed into heat, light, sound, electricity, &c. ; though in the transformation from one form of energy to another there is always an apparent loss of energy due to various causes, which is in fact a deterioration and not an actual loss.

138. We will consider this subject more fully later on ; our immediate object is to apply the principle to the machines considered in Chapter VIII.

In all the cases which we have there considered we have found the conditions of equilibrium, this being the primary object of statics. When there is equilibrium there is no motion, and consequently no work is done. We shall now imagine that there is a small displacement ; this is called a *virtual displacement*, since it is imaginary and cannot actually take place, and the work which would be done during such a displacement is called **virtual work**.

By the principle of the conservation of energy the total work done by a machine in motion is zero—that is, the work done by P is equal to the work done against W ; or, since the one is positive and the other negative, the algebraical sum of the work done is zero ; so in all machines which are at rest, **the algebraical sum of the virtual work done by the**

various forces during any virtual displacement must be zero.

We may now assume this principle, and from it derive the conditions of equilibrium ; or we may, by using the conditions of equilibrium already obtained in Chapter VIII., prove the principle in any particular case.

Note.—If the virtual displacement be perpendicular to the direction of any force, the virtual work done by the force is zero.

Hence, in solving problems by the principle of virtual work, if there is a force in the system the value of which we do not wish to ascertain, we shall suppose the displacement to take place in a direction $\perp^r$ to the line of action of this force, so that it will not then enter into the equation of virtual work.

Again, in certain cases we may have to consider more than one virtual displacement, in order that we may get several equations between the forces of the system, since one equation is often insufficient to completely determine the magnitude of each force.

139. Sometimes Aa —the projection of the displacement of the point of application on the line of action of F —is called the *virtual velocity* of the force F , and the product of $F \cdot Aa$ is called the *virtual moment* of the force.

Thus the principle stated at the end of the last article is called the principle of virtual velocities, and is stated thus : *the algebraical sum of the virtual moments of the various forces of a system in equilibrium is zero.*

140. *To prove the principle of virtual work in a lever of the second class.*

Let ABC be a lever of the second class, movable about the fulcrum C .

$$\text{Then } P \cdot AC = W \cdot BC.$$

Suppose that the points of application of A , B to be slightly displaced by the rotation of the lever through a small angle α .

Then AA' and BB' are the small arcs of circles whose centre is C ; but when α is indefinitely small they may be looked upon as straight lines perpendicular to ACB , and

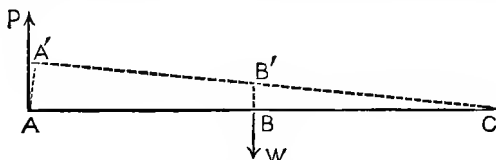


FIG. 107.

consequently represent the displacements of A and B in the lines of action of P and W .

Now $AA' = CA \cdot \alpha$ and $BB' = CB \cdot \alpha$.

Hence the algebraical sum of the virtual work of the forces

$$\begin{aligned} &= P \cdot AA' - W \cdot BB' \\ &= P \cdot CA \alpha - W \cdot CB \cdot \alpha \\ &= (P \cdot CA - W \cdot CB) \alpha \\ &= 0, \quad \text{since } P \cdot AC = W \cdot BC, \end{aligned}$$

which proves the principle.

Note.—The virtual work of the third force—namely, the reaction of the fulcrum—is zero, because there is no virtual displacement of the fulcrum.

It will be noticed that in the preceding example, as in those which follow, if work is supposed to be done *by* P it must be done *against* W , and *vice versa*.

$$\text{Since } \frac{\text{Displacement of } P}{\text{Displacement of } W} = \frac{CA}{CB} = \frac{W}{P}$$

the principle used to be stated as follows :

What is gained in power is lost in distance.

141. Assuming the principle of virtual work to be true in the case of a straight lever, when the forces P and W are not parallel, find the condition of equilibrium.

Let ACB be the lever, movable about the fulcrum C ; suppose A to receive a small displacement to A' , in conse-

quence of which B is moved to B', the lever being turned through a small angle α .

Draw A'a, CM perpendicular to the direction of P
and draw B'b, CN " " " W.

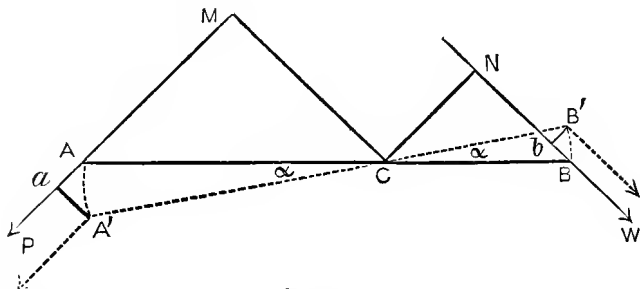


FIG. 108.

Then AA' and BB' are small arcs of circles whose radii are CA, CB respectively; hence

$$AA' = CA \cdot \alpha, \quad BB' = CB \cdot \alpha.$$

Since α is very small, AA', BB' may be considered to be straight lines, and the angles CAA', CA'A, CBB', CB'B to be right angles.

Hence the triangles AA'a, CAM are similar;

$$\therefore \frac{Aa}{AA'} = \frac{CM}{CA}.$$

$$\text{Now } AA' = CA \cdot \alpha; \therefore Aa = \frac{CM}{CA} \times CA \cdot \alpha = CM \cdot \alpha.$$

$$\text{Similarly} \quad Bb = \frac{CN}{CB} \times CB \cdot \alpha = CN \cdot \alpha.$$

Now, by the principle of virtual work,

$$P \cdot Aa - W \cdot Bb = 0;$$

$$\therefore P \cdot Aa = W \cdot Bb,$$

$$P \cdot CM \cdot \alpha = W \cdot CN \cdot \alpha;$$

$$\therefore P \cdot CM = W \cdot CN;$$

that is, the sum of the moments of P and W about C is zero.

142. *Apply the principle of virtual work to find the conditions of equilibrium on the wheel and axle,*

Suppose that there is a small displacement of the machine, in consequence of which it turns about its axis through a small angle α .

Then $P \cdot A a - W \cdot D d = 0$;

$$\begin{aligned}\therefore \frac{P}{W} &= \frac{D d}{A a} \\ &= \frac{O D \cdot a}{O A \cdot a} \\ &= \frac{O D}{O A};\end{aligned}$$

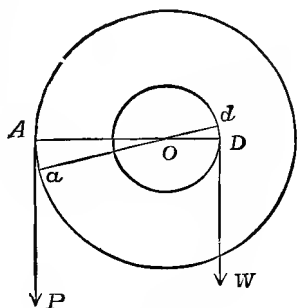


FIG. 109.

$\therefore P \times \text{radius of wheel} = W \times \text{radius of axle}.$

143. *Prove the principle of virtual work for the first system of pulleys when the weights of the pulleys are taken into account.*

Using the figure and notation of Article 121, let us suppose that W is raised through a small distance x , then A must also be raised through x ; to do this B must be raised through a distance $2x$, because the other end of the string round A is fixed; similarly C must be raised through a distance $2 \times 2x$, that is $2^2 \cdot x$, and so on. The n th pulley will have to be raised through a distance $2^{n-1} \cdot x$; consequently, P 's point of application must be moved through a distance $2^n \cdot x$.

Hence the sum of the virtual work

$$\begin{aligned}&= P \times 2^n x - W x - w_1 \cdot x - w_2 \cdot 2x - w_3 \cdot 2^2 x - \dots \\ &\quad - w_n \cdot 2^{n-1} x, \\ &= x (P \cdot 2^n - W - w_1 - 2w_2 - 2^2 w_3 - \dots - 2^{n-1} w_n) \\ &= x \times 0, \quad \text{by the equation of equilibrium,} \\ &= 0.\end{aligned}$$

144. *Apply the principle of virtual work to find the condition of equilibrium in the third system of pulleys.*

Using the figure and notation of Article 123, let us suppose that W is raised through a small distance x , then C

must descend through a distance x ; B will descend through a distance $2x$, on account of the descent of C, and also through a distance x on account of the ascent of W; that is, B will descend through a distance $2x + x$.

So A will descend through a distance $2(2x + x) + x$, that is, through $(2^2 + 2 + 1)x$; and if there are n pulleys the point of application of P will descend through a distance

$$(2^{n-1} + 2^{n-2} + \dots + 2 + 1)x;$$

that is, $\frac{2^n - 1}{2 - 1} \cdot x$, or $(2^n - 1)x$.

Now the algebraical sum of the virtual work is zero;

$$\therefore P \cdot (2^n - 1)x + w_1(2^{n-1} - 1)x + \dots + w_{n-1}(2 - 1)x - W \cdot x = 0;$$

$$\therefore P(2^n - 1) + w_1(2^{n-1} - 1) + \dots + w_{n-1} = W,$$

which is the equation of equilibrium.

145. *Prove the principle of virtual work in the case of a smooth inclined plane when the power acts along the plane.*

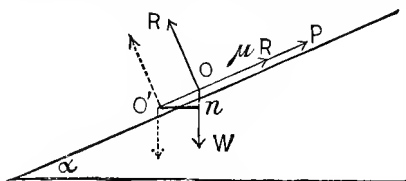


FIG. 110.

Suppose the point of application O to be displaced to O'; draw $O'n \perp^r$ to the vertical through O. Then, since OO' is $\perp^r$ to the direction

of R, the virtual work done by R during this displacement is zero.

Hence the algebraical sum of the virtual work

$$\begin{aligned} &= P \cdot OO' - W \cdot On \\ &= P \cdot OO' - W \cdot OO' \sin OO'n \\ &= P \cdot OO' - W \cdot OO' \sin \alpha \\ &= OO' (P - W \sin \alpha) \\ &= 0. \end{aligned}$$

We might have imagined a small displacement x in the

line of action of W , in which case the sum of the virtual work

$$\begin{aligned} &= W \cdot x - P \cdot x \sin \alpha - R x \cos \alpha \\ &= x(W - P \sin \alpha - R \cos \alpha) \\ &= x \cdot 0 = 0. \end{aligned}$$

146. Use the principle of virtual work to find the condition of equilibrium on a rough inclined plane when the body is on the point of moving down the plane, the power acting up the plane.

Suppose a small displacement OO' up the plane.

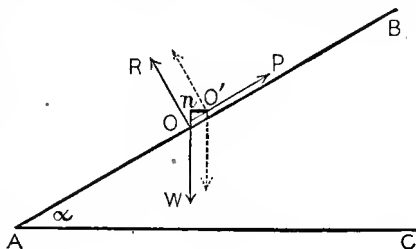


FIG. III.

Sum of the virtual work

$$\begin{aligned} &= W \cdot On - (P + \mu R) OO' = 0; \\ \therefore W \cdot OO' \sin \alpha &= (P + \mu R) OO'; \\ \therefore W \sin \alpha &= P + \mu R. \quad \dots \quad (1) \end{aligned}$$

Suppose now a small displacement $OO'' \perp^r$ to the plane.

Sum of the virtual work

$$\begin{aligned} &= R \cdot OO'' - W \cdot Om = 0; \\ \therefore R \cdot OO'' &= W(OO'' \cos \alpha); \\ \therefore R &= W \cos \alpha. \quad \dots \quad (2) \end{aligned}$$

Substitute in (1)

$$\begin{aligned} P + \mu W \cos \alpha &= W \sin \alpha; \\ \therefore P &= W(\sin \alpha - \mu \cos \alpha). \end{aligned}$$

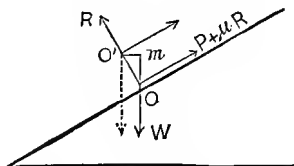


FIG. III2.

147. The preceding case may be solved more quickly thus:

Let S be the resultant pressure of the plane making an angle ϕ with the normal to the plane, so that $\tan \phi = \mu$.

Now suppose the virtual displacement to take place $\perp^r$ to the direction of S , as OO' .

Draw $O'n$, $O'm \perp$ to the directions of P and W respectively. Then, since sum of virtual work is zero,

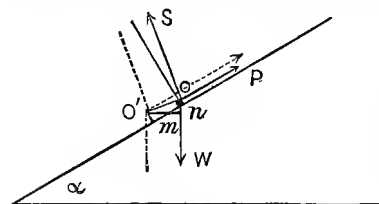


FIG. 113

$$W \cdot O'n - P \cdot O'm = 0.$$

Now OO' makes the same angle with the horizontal that S makes with the vertical, that is

$$\alpha - \phi;$$

$$\therefore \text{the angle } O'O'n = \alpha - \phi; \quad \therefore O'n = OO' \sin(\alpha - \phi);$$

$$\text{the angle } O'O'm = \phi; \quad \therefore O'm = OO' \cos \phi;$$

$$\therefore P \cdot OO' \cos \phi = W \cdot OO' \sin(\alpha - \phi);$$

$$\therefore P = W \frac{\sin(\alpha - \phi)}{\cos \phi}.$$

148. *To find the condition of equilibrium on a smooth screw by the principle of virtual work.*

(See figure at the end of Article 125.)

Suppose that A were to move completely round, then W would be raised through a distance equal to the vertical distance between two turns of the thread.

Hence, if displacement of $A = \frac{1}{n}$ (complete revolution of A)

the displacement of $W = \frac{1}{n} \left(\begin{array}{c} \text{vert. dist. between two turns of} \\ \text{the thread} \end{array} \right)$

Now P (P 's displacement) = W (W 's displacement);

$$\begin{aligned} \therefore \frac{P}{W} &= \frac{W\text{'s displacement}}{P\text{'s displacement}} \\ &= \frac{\text{vertical distance between two turns of thread}}{\text{circumference of circle described by } A}. \end{aligned}$$

149. *If any number of forces act at a point, the algebraical sum of the virtual work of all the forces = the virtual work of their resultant.*

Let O be the point, OO' the displacement, P any one of the forces making an angle θ with OO' .

Draw $O'a \perp$ to the direction of P , and suppose that R , the resultant of all the forces, makes an angle ϕ with OO' .

The sum of the virtual work of all the forces

$$\begin{aligned}
 &= \Sigma (P \cdot Oa) \\
 &= \Sigma (P \cdot OO' \cos \theta) \\
 &= OO' \cdot \Sigma (P \cos \theta) \\
 &= OO' \cdot R \cos \phi && (\text{Art. 55.}) \\
 &= R \cdot OO' \cos \phi \\
 &= \text{virtual work of the resultant.}
 \end{aligned}$$

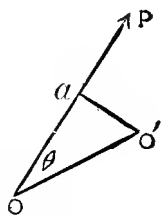


FIG. 114.

Hence, if the forces are in equilibrium, their resultant is zero, and consequently the sum of the virtual work of all the forces is zero.

For examples in this chapter the student may work the converse of any of the propositions in Articles 140 to 148.

PART II

KINEMATICS AND KINETICS

CHAPTER XI

KINEMATICS

THE student is here advised to read again the first seven articles of Chapter I.

We shall in the present chapter consider abstract motion without reference to the mass of the moving body, or the force which produces change of motion.

150. The rate at which a point moves is called its velocity, and, when uniform, is measured by $\frac{s}{t}$, where s is the space passed over in t seconds (Article 12).

Thus $v = \frac{s}{t}$ feet per second ;

$$\therefore s = v \cdot t.$$

151. *To find the measure of a velocity v when the units of space and time are altered.*

Suppose σ feet and τ seconds to be the new units of space and time.

The body describes v feet in one second,

or $\frac{v}{\sigma}$ (σ feet) in one second,

or $\frac{v\tau}{\sigma}$ (σ feet) in τ seconds ;

hence the new measure $= \frac{v\tau}{\sigma}$.

152. In Article 37 we have proved the parallelogram of velocities.

Hence all the results of Chapter III. which have been derived from the parallelogram of forces will hold *mutatis mutandis* for velocities.

Thus we shall have the following :

If V be the resultant velocity of two simultaneous velocities u and v , inclined to each other at an angle θ ,

$$V^2 = u^2 + v^2 + 2uv \cos \theta.$$

If ϕ be the angle this resultant makes with u ,

$$\tan \phi = \frac{v \sin \theta}{u + v \cos \theta}.$$

Triangle of Velocities.

If a point has simultaneously three velocities which can be represented in magnitude and direction by the sides of a triangle, taken in order, it will be at rest.

The **converse** is also true.

So also we have the **Polygon of Velocities**.

Similarly, the **resolved part** of a velocity v in a direction making an angle θ with the direction of v is $v \cos \theta$.

Again, the resultant of any number of simultaneous velocities may be found by resolving them all along two directions at right angles to each other, and finding the resultant of the algebraical sums of these two sets of components.

$$\text{Thus } V^2 = \{\Sigma (v \cos \theta)\}^2 + \{\Sigma (v \sin \theta)\}^2.$$

Change of Velocity.

153. When the velocity of a point changes in either magnitude or direction, or both, it is necessary for us to know what the change of velocity is.

Given therefore the velocity both before and after a certain change, we have to ask ourselves the following question : 'What velocity, if compounded with the original velocity, will

have a resultant velocity equal to the velocity after the change?'

Let OA represent the velocity of the point before the change, OB the velocity after. Join BA and complete the parallelogram $OACB$.

Then OA , OC represent two velocities whose resultant is represented by OB ; that is,

OC represents the velocity which must be communicated to a point in order to change its velocity from a velocity represented by OA to that represented by OB .

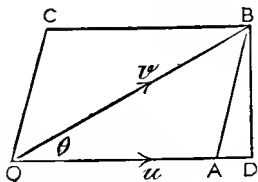


FIG. 115.

Hence OC or AB represents the change in velocity.

If $OA = u$, $OB = v$, and the angle $AOB = \theta$,

$$AB^2 = u^2 + v^2 - 2uv \cos \theta.$$

If ϕ be the angle AB makes with OA produced,

$$\tan \phi = \frac{BD}{AD} = \frac{v \sin \theta}{v \cos \theta - u}.$$

Example.—The velocity of a point is changed from 6 towards the east to $6\sqrt{2}$ towards the north-west. Find the change in velocity.

If V be the change in velocity, we have by the preceding article, putting $\theta = 135^\circ$,

$$\begin{aligned} V^2 &= u^2 + v^2 - 2uv \cos \theta \\ &= 36 + 72 - 2 \cdot 6 \cdot 6\sqrt{2} \left(-\frac{1}{\sqrt{2}}\right) \\ &= 108 + 72 = 180; \\ \therefore V &= 6\sqrt{5}. \end{aligned}$$

If ϕ be the angle V makes with the original velocity to the east,

$$\begin{aligned} \tan \phi &= \frac{v \sin \theta}{v \cos \theta - u} \\ &= \frac{6\sqrt{2} \times \frac{1}{\sqrt{2}}}{6\sqrt{2} \times \left(-\frac{1}{\sqrt{2}}\right) - 6} = \frac{6}{-12} = -\frac{1}{2}. \end{aligned}$$

This angle is greater than 150° .

a velocity $-u$. Complete the parallelogram $A'OBC$. Join OC .

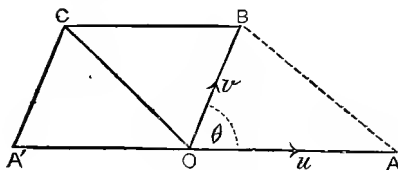


FIG. 116.

We may now suppose that the velocities of A are represented by OA , OA' , and that the velocities of B are represented by OB , OA' . Hence A is at rest.

Now OC represents the resultant of velocities OB , OA' ; that is, OC represents the velocity of B when A 's velocity is zero;

$\therefore OC$ or AB represents the velocity of B relative to A .

Similarly it may be shown that BA represents the velocity of A relative to B .

$$\begin{aligned}\text{Now } AB^2 &= AO^2 + OB^2 - 2AO \cdot OB \cos \theta \\ &= u^2 + v^2 - 2uv \cos \theta.\end{aligned}$$

This gives the magnitude.

If ϕ be the angle COA

$$\tan \phi = \frac{v \sin \theta}{v \cos \theta - u} \quad (\text{as in Article 153.})$$

The preceding result may also be conveniently arrived at in the following manner:

Let OB in the figure represent v , the velocity of B , as before. Resolve OB into two components, one of which is equal and parallel to u , the velocity of A , by completing the parallelogram $BAOC$.

OA , OC will therefore represent the whole velocity of B ; but the velocity OA is A 's velocity and produces no change in the distance between A and B ;

$\therefore OC$ represents the velocity of B relative to A .

Example 1.—Two points have equal velocities inclined to each other at an angle of 60° . Find the relative velocity,

Let the common velocity be v .

$$\begin{aligned} (\text{Relative velocity})^2 &= v^2 + v^2 - 2v^2 \cos 60^\circ \\ &= v^2 + v^2 - 2v^2 \left(\frac{1}{2}\right) \\ &= v^2; \end{aligned}$$

$\therefore$ relative velocity $= v$.

Hence the relative velocity is an equal velocity, and makes an angle of 120° with either velocity.

This result could also have been obtained at once from the geometry of the figure.

Example 2.—Two points A and B are distant a feet apart at a certain time. Their relative velocity is v , making an angle θ with A B. How near to each other can they get? Find also the time that elapses before they arrive at their nearest distance.

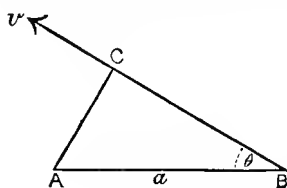


FIG. 117.

Suppose that A is brought to rest by giving each point an additional velocity equal and opposite to A's velocity; then if $\angle ABC = \theta$, BC represents the direction of the velocity of B relative to A. If AC be $\perp$ to BC, AC is the shortest distance between them.

$$AC = a \sin \theta.$$

If t be the time,

$$t = \frac{BC}{v} = \frac{a \cos \theta}{v}.$$

156. When a point is moving with variable velocity in a straight line, the rate at which the velocity changes is called its Acceleration, and is in the direction of motion.

It is then said to be uniform when there are equal increments of velocity in equal intervals of time. Uniform acceleration in the direction of motion is measured by $\frac{v}{t}$, where v is the velocity gained by the point in t seconds. (See Article 17.)

Thus, if a be the acceleration,

$$a = \frac{v}{t} \text{ units of velocity per second.}$$

When we say that the acceleration of a point in any particular direction is a , we mean that its velocity in that

direction is increased at the rate of a feet per second, per second.

Acceleration need not necessarily be in the direction of motion. Thus a point may have a uniform velocity in one direction and a uniform acceleration in another. The path of the point will then be a curve, and its velocity will be the resultant of two velocities—namely, the uniform velocity in the one direction, and the velocity due to the acceleration in the other. The rate of motion *along* the curve is not uniformly accelerated, but the rate of change of velocity is the given uniform acceleration.

In this case we consider the two motions separately, and the acceleration is measured by the change of velocity in the particular direction in which the acceleration takes place.

But so long as the acceleration remains uniform and in the same direction, it is measured by $\frac{v}{t}$; where v is the change of velocity *in that direction* in t seconds.

$$\text{Hence } v = at,$$

where v stands for the velocity gained in time t in the direction in which the acceleration a takes place.

Velocity may change in magnitude only, as in the case of an acceleration in the direction of motion; or it may change both in magnitude and direction, as in the case of an acceleration which is not in the direction of motion; or it may change in direction without changing in magnitude, as in the case of a ball tied to a string and made to revolve in a horizontal circle with uniform velocity.

At present we shall only deal with uniform acceleration in the direction of motion.

157. *To find the measure of an acceleration a when the units of space and time are altered.*

Suppose σ feet and τ seconds to be the new units of space and time.

The velocity of the point is accelerated by a feet per second in 1 second,

or $\frac{a}{\sigma}$ (σ feet) per second in 1 second,

or $\frac{a\tau}{\sigma}$ (σ feet) per τ seconds in 1 second,

or $\frac{a\tau^2}{\sigma}$ (σ feet) per τ seconds in τ seconds.

Hence the new measure of the acceleration is $\frac{a\tau^2}{\sigma}$.

158. *To find the unit of acceleration when the units of space and time are altered.*

Let σ feet and τ seconds be the new units of space and time.

The new unit of acceleration

= a velocity of σ feet per τ seconds in τ seconds

= σ (a velocity of 1 foot per τ seconds in τ seconds)

= $\frac{\sigma}{\tau}$ (a velocity of 1 foot per 1 second in τ seconds)

= $\frac{\sigma}{\tau^2}$ (a velocity of 1 foot per second in 1 second)

= $\frac{\sigma}{\tau^2}$ (the ordinary unit).

Since accelerations can also be compounded by the parallelogrammic law, it follows that all that has been said in Article 152 about velocities holds true also for accelerations.

159. *Lemma. The average of any number of terms in arithmetical progression is equal to the arithmetic mean between the first and last—that is, half the sum of the first and last.*

Let a be the first term, d the common difference, n the number of terms, l the n th term, S the sum of all the terms.

Then $l = a + (n - 1)d$,

$$S = \frac{n}{2} (a + l).$$

$$\text{Average of all the terms} = \frac{S}{n} = \frac{\frac{n}{2} (a + l)}{n} = \frac{a + l}{2}.$$

160. *When a point is moving with a constant acceleration a in the direction of motion, the velocities at the beginning of each second form a series in arithmetical progression.*

For if u be the velocity at any time,

the velocity after 1 second is $u + a$,

„ 2 seconds is $u + 2a$,

„ 3 „ „ $u + 3a$, and so on ;

„ t „ „ $u + ta$,

and these form a series in arithmetical progression.

Hence, if v be the velocity at the end of time t ,

$$v = u + at \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\text{The average velocity} = \frac{u + v}{2} = \frac{u + u + at}{2}$$

$$= u + \frac{t}{2} \cdot a$$

= velocity at the middle of the time.

161. It must be borne in mind that the velocity is not constant during any second, but is always changing uniformly. This does not affect the result of the last article.

For we may suppose that each second is divided up into n very small equal intervals of time, during each of which we may suppose that the velocity is constant, and at the end of each of which we may suppose the velocity to be increased by $\frac{a}{n}$.

Thus during each second the velocities at the end of each of these small intervals will form an arithmetical series whose constant difference is $\frac{a}{n}$.

Thus the average velocity during the first second

$$= \frac{u + u + a}{2}$$

$$= u + \frac{1}{2} a$$

= velocity at the middle of the second,

and so on.

The average velocity during the t th second

$$= \frac{u + (t-1)a + u + ta}{2}$$

$$= u + (t - \frac{1}{2}) a$$

= velocity at the middle of the t th second.

Hence the average velocity during the whole time

= the average of all the averages

$$= \frac{u + \frac{1}{2}a + u + (t - \frac{1}{2})a}{2}$$

$$= u + \frac{t}{2} \cdot a.$$

162. To find the space passed over by a point in t seconds, which starts with an initial velocity u , and has a constant acceleration a in the direction of motion.

Let s = space passed over.

Since the point is moving with constant acceleration, the average velocity during the time is equal to the velocity at the middle of the time; and hence the space passed over in time t is the same as would be described by a point moving with a uniform velocity equal to the velocity at the middle of the time;

$$\begin{aligned}\therefore s &= (\text{average velocity}) \times t \\ &= \left(u + \frac{t}{2} \cdot a\right) t \\ &= ut + \frac{1}{2}at^2 \quad \dots \dots \dots (2)\end{aligned}$$

To find the relation between v , a , and s , eliminate t . The above equation may also be written in the form

$$s = \left(\frac{u+v}{2}\right) \cdot t.$$

But $v = u + at$ (1) (Art. 160); $\therefore t = \frac{v-u}{a}$.

Hence $s = \left(\frac{v+u}{2}\right) \left(\frac{v-u}{a}\right) = \frac{v^2 - u^2}{2a}$

$$v^2 - u^2 = 2as,$$

or

$$v^2 = u^2 + 2as \quad \dots \dots \dots (3)$$

Equation (3) may also be obtained by substituting the value of t from (1) in (2).

Thus $s = u \left(\frac{v-u}{a}\right) + \frac{1}{2}a \left(\frac{v-u}{a}\right)^2;$

$$\begin{aligned}\therefore 2as &= 2u(v-u) + (v-u)^2 \\ &= v^2 - u^2.\end{aligned}$$

163. We now have the three equations—

$$v = u + at \quad . \quad . \quad . \quad . \quad (1)$$

$$s = ut + \frac{1}{2}at^2 \quad . \quad . \quad . \quad . \quad (2)$$

$$v^2 = u^2 + 2as \quad . \quad . \quad . \quad . \quad (3)$$

When u and a are given

(1) gives us the velocity at the end of a given time ;

(2) gives us the space described in a given time ;

(3) gives us the velocity after describing a given space.

If $u = 0$, the body starts from rest, and we have—

$$v = at,$$

$$s = \frac{1}{2}at^2,$$

$$v^2 = 2as.$$

If a is opposite to the direction of motion—that is, a retardation—put $-a$ for a in the above equations, and we have—

$$v = u - at,$$

$$s = ut - \frac{1}{2}at^2,$$

$$v^2 = u^2 - 2as.$$

Note.—If $a = 0$ the equations become

$$v = u, \quad s = ut, \quad v^2 = u^2.$$

164. It will be interesting to obtain equation (2) of the previous article by the *graphical* method.

The space described by a point moving uniformly is = velocity $\times$ time. If, therefore, we represent time by a straight line in one direction, and velocity by a straight line at right angles to it, the space described may be represented by a rectangle whose base represents the time and whose height represents the velocity.

Thus, if $ABCD$ is a rectangle whose base AB contains t units of length, and height AD u units of length, the area = ut , and represents the space described by a point in time t moving with uniform velocity u .

Now suppose the point to have an acceleration a in addition to the uniform velocity u .

Produce BC to E , making $CE = ta$, so that $BE = u + ta$ = the velocity at the end of time t . Join DE .

To find the velocity at any time represented by AP .

Draw $Pp \perp AB$, cutting DE in p . Pp shall represent the velocity after time AP .

Let $AP = n$ seconds, where n may be integral or fractional, and let Pp cut DC in p' .

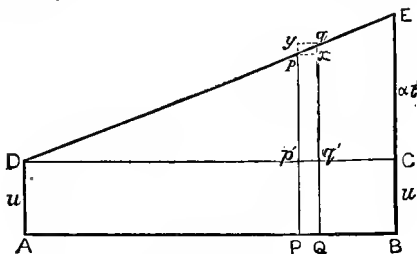


FIG. 118.

Then, by similar triangles,

$$\frac{p p'}{E C} = \frac{D p'}{D C} = \frac{A P}{A B} = \frac{n}{t};$$

$$\therefore p p' = \frac{n}{t} (a t) = n a.$$

Hence $Pp = u + na$ = velocity after n seconds. Similarly, the velocity at any other time will be given by the ordinate drawn from the point in AB which represents the time, to cut the line DE .

Let Q represent a time very near to P , so that PQ represents a small interval of time during the body's motion; the ordinate Qq represents the velocity at Q , and during the time PQ the velocity increases uniformly from Pp to Qq .

Draw px parallel to PQ to cut Qq in x , and qy parallel to PQ to cut Pp in y .

Then, if the velocity during the time PQ were constant and equal to the velocity at P , the space described during the time PQ would be represented by the rectangle $PQxp$; so also, if the velocity during the time PQ were constant and equal to the velocity at Q , the space described would be represented by the rectangle $PQqy$.

But the velocity is not constant, but changes from Pp to Qq : hence the actual space described is represented by an area which lies between the rectangles $PQxp$ and $PQqy$.

But in the limit where the time PQ is very small, the areas of these two rectangles are very nearly equal to each other and also to the area $PQqy$ which lies between them; and if PQ be taken small enough, these three areas are all equal. Hence the actual

space described during the time PQ is represented by the area $PQqp$.

It follows therefore that the space described during the time t or AB is represented by the area $ABED$;

$$\begin{aligned}\therefore s &= \text{area } ABED \\ &= \text{area of rectangle } AC + \text{area of triangle } DCE \\ &= AB \cdot BC + \frac{1}{2} DC \cdot CE \\ &= ut + \frac{1}{2} \cdot t \cdot at \\ &= ut + \frac{1}{2} at^2.\end{aligned}$$

165. In the previous article AB is called the time-line and DE the *curve of velocity*. In the case we have considered DE is a straight line, because the velocity is uniformly accelerated in the direction of motion; if this were not the case DE would no longer be a straight line, but an irregular line or a curve. This curve is of such a nature that if AT be taken to represent any time on the time-line, and and the ordinate Tt be drawn to cut the curve in t , then Tt represents the velocity at the time represented by AT .

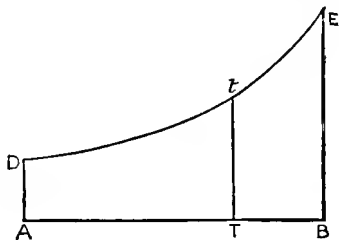


FIG. 119.

Then, whether the acceleration be uniform or not, the space described during any time represented by AB on the time-line will be represented by the area $ABED$, which is inclosed between the time-line, the curve of velocity, and the ordinates at the beginning and end of the time-line.

The method used in the preceding article is of the very greatest importance in the more advanced parts of the subject.

166. *To find the space described by a point whose velocity is uniformly accelerated in any particular second.*

Let u = initial velocity, a = the acceleration, then

$$s = ut + \frac{1}{2} at^2.$$

$$\begin{aligned}
& \text{The space described in the } t\text{th second} \\
&= \text{space described in } t \text{ seconds} - \text{space described in } (t-1) \text{ secs.} \\
&= ut + \frac{1}{2} a t^2 - u(t-1) - \frac{1}{2} a (t-1)^2 \\
&= u(t-t+1) + \frac{1}{2} a (t^2 - t^2 + 2t - 1) \\
&= u + \frac{1}{2} a (2t - 1) \\
&= u + a(t - \frac{1}{2}).
\end{aligned}$$

This might have been obtained more quickly by the following consideration :

$$\begin{aligned}
& \text{Space described in the } t\text{th second} \\
&= \text{space described in 1 second by a point moving with uniform} \\
& \quad \text{velocity equal to the velocity at the middle of the } t\text{th} \\
& \quad \text{second} \\
&= u + (t - \frac{1}{2}) a.
\end{aligned}$$

Putting $t = 1, 2, 3$, &c., we have—

$$\begin{array}{llll}
\text{space described during 1st second} & = & u + \frac{1}{2} a, \\
\text{„ „ 2nd „} & = & u + \frac{3}{2} a, \\
\text{„ „ 3rd „} & = & u + \frac{5}{2} a.
\end{array}$$

Hence the spaces described in consecutive seconds form a series in arithmetical progression whose constant difference is a .

If a body starts from rest subject to a constant acceleration, the velocity at any time varies as the time, and the space passed over varies as the square of the time.

$$\text{For } v = at, \text{ and } s = \frac{1}{2} at^2.$$

Now a is constant ;

$$\therefore v \propto t, \text{ and } s \propto t^2.$$

167. *A body is projected with a velocity u , and is subject to a uniform retardation a opposite to the direction of motion. To prove that the time that elapses before it comes to rest is equal to the time it takes to return to the point of projection, and that the velocity with which it passes the point of projection on its return is equal to the velocity with which it was projected.*

The equations which determine the motion are—

$$v = u - at \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$s = ut - \frac{1}{2}at^2 \quad . \quad . \quad . \quad . \quad (2)$$

$$v^2 = u^2 - 2as \quad . \quad . \quad . \quad . \quad (3)$$

To find the time that has elapsed when it comes to rest, put $v = 0$ in equation (1),

$$0 = u - at; \therefore t = \frac{u}{a}.$$

To find the time that has elapsed when it is again at the starting point, put $s = 0$ in equation (2),

$$0 = ut - \frac{1}{2}at^2;$$

$$\therefore t(u - \frac{1}{2}at) = 0; \therefore t = 0, \text{ or } \frac{2u}{a};$$

that is to say, it is at the starting point when it starts, and again after a time $\frac{2u}{a}$, which is twice the time that elapsed before it came to rest; hence, time taken to go out = time taken to return.

To find the velocity with which it returns to the starting point, put $s = 0$ in equation (3),

$$v^2 = u^2;$$

$$\therefore v = \pm u;$$

that is to say, at the starting point the velocity is u when it starts and $-u$, or u in the opposite direction, when it returns.

Similarly, the velocity with which it passes any point on its way back is equal to the velocity with which it passed it on its way out. For, put $s = a$ in equation (3), then

$$v^2 = u^2 - 2a \cdot a;$$

$$\therefore v = \pm \sqrt{u^2 - 2aa}.$$

The positive value denotes the velocity on the way out; the negative value is the velocity at the same place on the way back.

168. If a body has an acceleration which is not in the direction of motion, its path is a curve on which we can find points in the following manner.

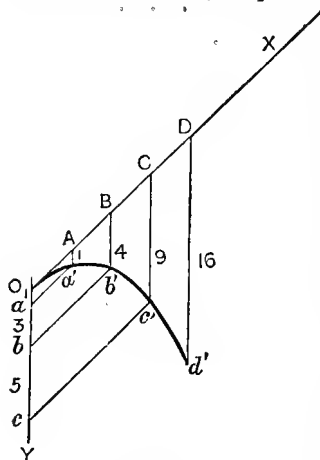


FIG. 120.

Suppose u the uniform velocity with which it starts from O in the direction Ox , and a the uniform acceleration in the direction Oy .

Let OA represent u ; then if there were no acceleration the body would be at the points A, B, C , &c., at the end of consecutive seconds.

But in consequence of the acceleration a in direction Oy it also describes distances parallel to Oy , which distances vary as the square of the time which has elapsed.

Thus at the end of 1 second $OA = \frac{1}{2}at$,
 " " 2 seconds $OB = \frac{1}{2}at^2$,
 " " 3 " $OC = \frac{1}{2}at^3$,

and so on.

These two motions are quite independent of each other and may be considered separately. (Art. 34.)

Thus at the end of consecutive seconds the body will be at the points a', b', c', d' , &c., and its path will be the curved line $Oa'b'c'd'$. The path of the body is in fact the same as if it were projected with uniform velocity u along a smooth tube Ox while the tube moved with uniform acceleration a in the direction Oy .

To find the velocity after any time t .

Let θ be the angle between Ox and Oy ,

the velocity in direction $Ox = u$,

" " $Oy = at$;

$$\therefore (\text{velocity})^2 = u^2 + (at)^2 + 2uat \cos \theta.$$

The direction of this resultant velocity can be found as in

Article 46. This direction is the actual direction in which the body is moving at the time, and is a tangent to the curve at the point which represents the body's position at that time. This curve is a parabola.

Note.—It is found that the path of the body is a curve when the two motions are of *different* kinds. If the body started from rest at O with two uniform accelerations, its path would be a straight line by the parallelogram of accelerations.

Example 1.—The measure of an acceleration in the foot-second units is 3. The units of space and time being altered, the measure of the acceleration is 8. If the new units of space be 2 yards, what is the new unit of time?

Let t seconds be the new unit of time.

The acceleration

$$= 8 \text{ (2 yards) per } t \text{ seconds in } t \text{ seconds}$$

$$= 48 \text{ feet per } t \text{ seconds in } t \text{ seconds}$$

$$= \frac{48}{t} \text{ feet per second in } t \text{ seconds}$$

$$= \frac{48}{t^2} \text{ feet per second in 1 second;}$$

$$\therefore 3 = \frac{48}{t^2},$$

$$\text{or} \quad 3t^2 = 48,$$

$$\text{or} \quad t^2 = 16;$$

$$\therefore t = 4.$$

Hence the new unit of time is 4 seconds.

Example 2.—A velocity of $\frac{1}{2}$ and an acceleration of $\frac{1}{10}$ in the foot-second units become 10 and 120 when the units of space and time are altered. Find the new units of space and time.

Let s feet and t seconds be the new units.

A velocity of $\frac{1}{2}$ foot per second

$$= \text{a velocity of } \frac{t}{2s} (s \text{ feet}) \text{ per } t \text{ seconds;}$$

$$\therefore \frac{t}{2s} = 10 \quad \dots \dots \dots (1)$$

An acceleration of $\frac{1}{10}$ foot per second per second

$$= \text{an acceleration of } \frac{t^2}{10s} (s \text{ feet}) \text{ per } t \text{ seconds per } t \text{ seconds;}$$

$$\therefore \frac{t^2}{10s} = 120 \quad \dots \dots \dots (2)$$

Substitute from (1) in (2)

$$(20s)^2 = 1200s;$$

$$\therefore s = 3$$

and hence $t = 60$;

$\therefore$ the new units of space and time are 1 yard and 1 minute.

Example 3.—A point moving with uniformly accelerated velocity has a velocity of 3 feet per second; 5 seconds later its velocity is 10. Find the space passed over in the time, and the acceleration.

Space described in the time = space that would be described by a body moving with uniform velocity equal to the velocity at the middle of the time

$$= \frac{3+10}{2} \cdot 5$$

$$= \frac{65}{2} = 32\frac{1}{2} \text{ feet.}$$

$$\text{Acceleration} = \frac{\text{change in velocity}}{\text{time}}$$

$$= \frac{7}{5} \text{ feet per second per second.}$$

To work it by our formulæ :

$$u = 3, \quad v = 10, \quad t = 5;$$

substitute in the equation $v = u + at$,

$$\text{whence } a = \frac{7}{5}.$$

To find s , substitute in either of the equations

$$s = \left(\frac{u+v}{2} \right) t,$$

or

$$s = ut + \frac{1}{2} at^2.$$

Example 4.—A body has an initial velocity 20, and moves with constant acceleration 3 opposite to the direction of motion. Find

- (1) its velocity after 5 seconds,
- (2) space passed over in 5 seconds,
- (3) velocity when it is 56 feet away,
- (4) time which has elapsed when it is 56 feet away,
- (5) the distance traversed before it stops,
- (6) the time which elapses before it stops,
- (7) its position after 14 seconds.

Here $u = 20$, $a = -3$.

$$(1) \quad v = u + at \\ = 20 - 3 \cdot 5 = 5.$$

$$(2) \quad s = ut + \frac{1}{2}at^2 \\ = 20 \cdot 5 - \frac{3}{2}(25) \\ = 100 - 37.5 = 62\frac{1}{2}.$$

$$(3) \quad v^2 = u^2 + 2as, \\ v^2 = (20)^2 - 6 \cdot 56 \\ = 400 - 336 = 64; \\ \therefore v = \pm 8.$$

$$(4) \quad s = ut + \frac{1}{2}at^2, \\ 56 = 20t - \frac{3}{2}t^2, \\ 3t^2 - 40t + 112 = 0 \\ (t - 4)(3t - 28) = 0; \\ \therefore t = 4, \text{ or } 9\frac{1}{3}.$$

$$(5) \quad v^2 = u^2 + 2as;$$

put $v = 0$,

$$0 = (20)^2 - 6s; \\ \therefore s = \frac{400}{6} = 66\frac{2}{3}.$$

$$(6) \quad v = u + at;$$

put $v = 0$,

$$0 = 20 - 3t; \\ \therefore t = \frac{20}{3} = 6\frac{2}{3}.$$

This result must obviously lie half-way between the two answers to (4).

$$(7) \quad s = ut + \frac{1}{2}at^2 \\ = 20 \times 14 - \frac{3}{2}14^2 \\ = 280 - 3 \cdot 98 \\ = -14;$$

that is to say, it will have returned past the place where it started and have reached a place 14 feet on the other side.

Examples.

1. Find the actual velocity of a particle which has two independent velocities of 5 and 3 in directions making an angle of 60° with each other.

For other examples on the parallelogram of velocities take any of the examples 5 to 9, 12 to 15, 21 to 24 at the end of Chapter III, substituting 'velocity' for 'force'.

2. A distance of 1,100 yards is described by a point moving with a uniform velocity of a mile an hour in the time $2\frac{1}{2}$. What is the unit of time?

3. Express a velocity of 1 mile an hour in terms of a unit velocity of 10 feet a second.

4. Express the acceleration 32 feet per second per second in terms of the units 1 yard and 1 minute.

5. If 1 inch, 1 second, and 1 ounce be the units of length, time, and mass respectively, find the measure of the force equal to the weight of 1 pound.

6. A stone is projected horizontally with a velocity of 16 feet per second from a steamer which is moving at the rate of 12 miles per hour in a direction perpendicular to that in which the stone is projected. Find the initial velocity of the stone. Find also the velocity at the end of 1 second if the stone also acquires a vertical velocity of 32 feet per second in 1 second.

7. A vessel sailing due north at the rate of 7 miles an hour is under the influence of an easterly current running 3 feet a second. What is the actual velocity of the vessel, and how far will she be carried out of her course in 11 hours?

8. A train is moving 40 miles an hour when a ball is thrown horizontally out of the window in a direction at right angles to that of the train with a velocity relative to the train of 44 feet per second. What is the actual velocity of the ball at the moment of projection?

9. The unit of length is 5 feet, and the unit of time 6 seconds. How many units of velocity are there in the speed of a train going 10 miles an hour?

10. If a mile per minute be the unit of velocity, and 32 feet per second per second the unit of acceleration, find the units of space and time.

11. If the acceleration of a particle be 32 in terms of the units 1 foot and 1 second, what will be the measure of the acceleration when a mile is the unit of space and a minute the unit of time?

12. The unit of length is 6 feet, and the unit of time 5 seconds. What number will represent the acceleration of a train which acquires a velocity of 10 miles an hour in one minute from the start, the acceleration being assumed constant?

13. Find the velocity which would be acquired from rest by a

particle moving with a constant acceleration of 3 feet per second per second after traversing 22 miles.

14. A bucket is ascending the shaft of a mine at a uniform rate of 10 feet per second. When it is 50 feet from the top the tension of the cord is diminished, so that it now ascends with a velocity which is uniformly retarded, and comes to rest at the top. What is the retardation?

15. If the spaces described by a particle moving with uniform acceleration be 168 feet and 288 feet in the first three and the first four seconds respectively, find the initial velocity and the acceleration.

16. A point moving from rest with uniform acceleration acquires in 3 seconds a velocity of 1,000 yards a minute. In what time will it acquire a velocity of 100 miles an hour?

17. A body which was moving at the rate of 140 yards per minute increased its velocity uniformly, so that at the end of a quarter of a minute it was moving at the rate of 15 miles an hour. What was its acceleration?

18. A particle acquires a velocity of 1,000 miles an hour in 22 minutes. Compare its acceleration with that of gravity.

19. A particle moving with uniform acceleration a passes over a space s in time τ . Prove that its velocity at the beginning of time τ is equal to $\frac{s}{\tau} - \frac{a\tau}{2}$.

20. A train moving from rest with uniform acceleration attains a velocity of 20 miles an hour in 1 minute. Find its acceleration, adopting a foot and a second as the units.

21. A train moving from rest with uniform acceleration acquires after three minutes a velocity of 30 miles an hour. Find the distance travelled.

22. In what time would a body acquire a velocity of 225 miles an hour if the acceleration were 33 feet per second per second?

23. Find the mean acceleration of a train which increases its speed from 40 to 49 miles per hour while descending an incline in $4\frac{1}{2}$ minutes.

24. In the above question, if the acceleration due to gravity be 32, determine the slope of the incline.

25. A train is travelling at the rate of 44 feet per second on

level rails, and it is observed that when steam is shut off the train stops after travelling a quarter of a mile further; to what retardation is it subject? If the mass of the train be 1 ton, find the resistance in poundals. If this resistance be all due to the friction of the rails, find the coefficient of friction.

26. A body is moving with uniformly accelerated velocity. Compare the spaces passed over in the n th and $(n + 1)$ th seconds after the motion commenced.

27. If a ball with a velocity of 900 feet per second enters a block of wood to the depth of 9 inches, what velocity will produce a penetration of 16 inches, the resistance of the wood being supposed uniform?

28. Using only the formulæ for accelerated motion, determine with what velocity a stone will strike the ground which falls through a height of 257.6 feet in 4 seconds.

29. A body observed three times at intervals of a second is found to have velocities 36, 60, 84. Find the spaces passed over in the intervals, the acceleration being uniform.

30. An engine-driver on passing through a station reduces speed from 60 miles an hour to 40 miles an hour in the space of 400 yards. What is the retardation?

31. A train moving with a constant retardation travels 87 feet in the first second and 85 in the next. How far will it go before it comes to rest?

32. A train starting from rest travels 1 foot in the first second, 3 feet in the next, 5 feet in the next, and so on. How long will it be before it attains a velocity of a mile a minute?

33. A point moving with uniform acceleration describes 52 feet in the half second which elapses after the third second of its motion. Compare its acceleration with that of a falling body.

34. A uniformly accelerated body passed two points 6 feet apart in one third of a second; four seconds after reaching the first of these points the body had a velocity of 110 feet per second. Find the distance travelled in one second after reaching the second point.

35. A passenger in a train moving with uniform acceleration observed that in one minute and a quarter the train travelled a mile. He waited a minute and then found that the train took

another minute to travel the next mile. What was the velocity at the end of the last minute and what was the acceleration?

36. It is observed that as a vessel is steaming 10 knots an hour due N., while the wind is blowing from E.N.E., the direction of the smoke track is from N.E. to S.W. Show by a diagram how you could ascertain the velocity of the wind.

37. A ship is steaming due N. with a speed of 15 miles per hour, and another due E. with a speed of 20 miles per hour. The second passes the point of intersection of the two tracks 1 hour before the first reaches it. Find the least distance between the ships.

CHAPTER XII

MOTION DUE TO THE FORCE OF GRAVITY

169. THE measure of the force of gravity on a body is called the body's weight (Article 7).

On account of this force, we find that when a body is allowed to fall freely *in vacuo* it moves vertically downwards with a constant acceleration, which we denote by g .

Since the earth is not a perfect sphere, the force of gravity varies somewhat at different points on the earth's surface; hence the value of g varies slightly. Its value at the Equator is about $32\cdot08$, and at the Poles $32\cdot25$ when measured in feet and seconds.

In England its value is approximately $32\cdot2$, or, more accurately, $32\cdot18$.

In working numerical examples, its value may be taken to be 32 , unless it is otherwise stated.

Thus g = an acceleration of velocity vertically downwards of 32 feet per second per second.

This value of g only applies to the motion *in vacuo*.

In nature the resistance of the air tends to retard the velocity. This retardation will not be taken into account.

170. The formulæ of the preceding chapter may now be used for vertical motion under the action of gravity by putting $a = g$ when the motion is measured vertically downwards.

Thus

$$\begin{aligned} v &= u + g t, \\ s &= u t + \frac{1}{2} g t^2, \\ v^2 &= u^2 + 2 g s. \end{aligned}$$

If the body starts from rest $u = 0$, and we have

$$\begin{aligned}v &= g t, \\s &= \frac{1}{2} g t^2, \\v^2 &= 2 g s.\end{aligned}$$

If the body is projected vertically upwards with a velocity u we put $a = -g$, and we have for the upward motion

$$\begin{aligned}v &= u - g t, \\s &= u t - \frac{1}{2} g t^2, \\v^2 &= u^2 - 2 g s.\end{aligned}$$

If we use these latter equations in solving numerical examples, a negative answer for v means that the body is on its way down, and a negative answer for s means that the body is below the point from which its motion is considered.

171. *A body is projected vertically upwards with a velocity u . To find the greatest height to which it will go, and the time it takes to reach this height.*

When it arrives at its greatest height its velocity will be zero ; hence we put $v = 0$ in the equation,

$$\begin{aligned}v^2 &= u^2 - 2 g s, \\0 &= u^2 - 2 g s ; \\\therefore s &= \frac{u^2}{2 g}.\end{aligned}$$

To find the time, put $v = 0$ in the equation,

$$\begin{aligned}v &= u - g t, \\0 &= u - g t ; \\\therefore t &= \frac{u}{g}.\end{aligned}$$

Also, as in Article 167 of the last chapter, the time it takes to go up is equal to the time it takes to come down ; the velocity with which it returns to the earth is equal to the velocity with which it was projected ; the velocity with which it passes any point on its way down is equal to the velocity with which it passed the same point on its way up ; the time between any two points on the way up is equal to the time between the same two points on the way down.

Example 1.—A man stands at the bottom of a cliff 100 feet high; with what velocity must he throw a stone up in order that it may just reach the top?

$$\begin{aligned} v^2 &= u^2 - 2gs; \\ \text{put } v &= 0, \quad s = 100, \\ 0 &= u^2 - 64 \times 100; \end{aligned}$$

$$\therefore u^2 = 6400,$$

or

$$u = 80 \text{ feet per second.}$$

The time taken by the stone to get there = $2\frac{1}{2}$ seconds.

Example 2.—A stone is projected vertically upwards, and on its way up passes a point 640 feet above the ground with a velocity of 96. Find the time which elapses before it reaches the ground.

Here we will consider that s is measured upwards from the point 640 feet above the ground, and substitute in the equation

$$\begin{aligned} s &= ut - \frac{1}{2}gt^2, \\ -640 &= 96t - 16t^2, \\ t^2 - 6t - 40 &= 0 \\ (t - 10)(t + 4) &= 0; \\ \therefore t &= 10, \text{ or } -4. \end{aligned}$$

Hence the stone will reach the ground 10 seconds after it passes the point 640 feet above the ground.

The negative answer means that the stone started from the ground 4 seconds before it reached the above point.

The answer of 10 seconds is thus made up as follows :—

- 3 seconds from the given point to the highest point,
- 3 seconds back to the given point,
- 4 seconds from the given point to the ground.

Example 3.—A stone is dropped down a perpendicular cliff h feet high. At the same time another stone is thrown up from the bottom with a velocity which would just enable it to reach the top. When and where will they meet?

Let t be the time after which they will meet, x the distance from the top, u the velocity with which the stone is projected from the bottom.

Then

$$\begin{aligned} u^2 &= 2g \times h, \\ x &= \frac{1}{2}gt^2, \\ h - x &= ut - \frac{1}{2}gt^2; \end{aligned}$$

adding, we get $h = ut$;

$$\begin{aligned}\therefore t &= \frac{h}{u} \\ &= \frac{h}{\sqrt{2gh}} = \sqrt{\frac{h}{2g}} \\ &= \frac{1}{2} \sqrt{\frac{2h}{g}} = \frac{1}{2} \text{ (time required to fall } h \text{ feet);} \\ x &= \frac{1}{2} g t^2 = \frac{1}{2} g \left(\frac{h}{u} \right)^2 \\ &= \frac{1}{2} g \frac{h^2}{2gh} = \frac{1}{4} h.\end{aligned}$$

Thus, if the cliff be 300 feet high, $t = 2.165$ seconds and $x = 75$ feet.

Example 4.—A body is projected vertically upwards with a velocity $5g$; three seconds afterwards another body is projected vertically upwards with a velocity $4g$. When and where will they meet?

Let x = distance from the ground,

t = the time after the first body starts,

$x = 5gt - \frac{1}{2}gt^2$ for the first body,

$x = 4g(t-3) - \frac{1}{2}g(t-3)^2$ for the second body.

Equate these values of x :

$$\begin{aligned}5t - \frac{1}{2}t^2 &= 4(t-3) - \frac{1}{2}(t-3)^2, \\ 10t - t^2 &= 8t - 24 - t^2 + 6t - 9, \\ 4t &= 33, \\ t &= 8\frac{3}{4} \text{ seconds;} \\ x &= 5g\left(\frac{33}{4}\right) - \frac{1}{2}g\left(\frac{33}{4}\right)^2 \\ &= \frac{33g}{4}\left(5 - \frac{33}{8}\right) \\ &= \frac{231g}{32} = 231 \text{ feet from the ground.}\end{aligned}$$

The answer shows that the bodies will not *meet*, but that the first body will *pass* the second when they are both on their way down $1\frac{3}{4}$ second before the first body reaches the ground, and $2\frac{3}{4}$ seconds before the second body reaches the ground.

Motion on a Smooth Inclined Plane.

172. A particle slides down a smooth inclined plane of given length ; to find the time it takes and the velocity acquired.

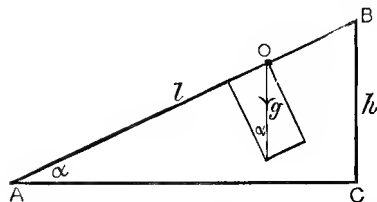


FIG. 121.

Let ABC be the plane, of inclination α . Let $AB = l$, and $BC = h$, and suppose the particle to be at O .

Now, if the plane were not there, the particle would fall freely to the ground with acceleration g , but this it cannot do ; we therefore resolve g parallel to the plane and perpendicular to the plane. The component $g \cos \alpha$ perpendicular to the plane has no effect upon the motion of the particle on the plane ; hence the total acceleration in the direction in which the particle can move is $g \sin \alpha$ down the line of greatest slope.

If the particle starts from rest at B we have

$$s = BA = \frac{1}{2} g \sin \alpha \cdot t^2 ;$$

$$\therefore t^2 = \frac{2 BA}{g \sin \alpha} = \frac{2 l}{g \sin \alpha}.$$

To find the velocity,

$$v^2 = 2 g \sin \alpha \cdot s = 2 g \sin \alpha \cdot BA.$$

$$\text{Now, } \sin \alpha = \frac{BC}{BA} ; \therefore v^2 = 2 g \cdot BC = 2 g h.$$

That is to say, **the velocity acquired in sliding down a smooth inclined plane is the same as that acquired by a body falling freely through a distance equal to the height of the plane.**

This result is of the greatest importance, for a curve may be looked upon as a series of inclined planes ; hence, the velocity acquired in sliding down a smooth curve can be proved to be equal to that which would be acquired in falling freely through the same vertical height.

Similarly, if a particle be projected up an inclined plane (α) with initial velocity u , its motion will be determined by the equations

$$\begin{aligned}v &= u - g \sin \alpha \cdot t, \\l &= u t - \frac{1}{2} g \sin \alpha \cdot t^2, \\v^2 &= u^2 - 2 g \sin \alpha \cdot l, \\&= u^2 - 2 g h.\end{aligned}$$

The velocity of projection that it should just reach the top
 $= \sqrt{2 g h}.$

In this case the time up the plane

$$= \frac{u}{g \sin \alpha} = \sqrt{\frac{2 g h}{g^2 \sin^2 \alpha}} = \sqrt{\frac{2 l}{g \sin \alpha}}.$$

173. *If chords be drawn through the highest point of a vertical circle, the time of descent down any such chord of a particle which starts from rest at the top is the same for all the chords, and is equal to the time down the vertical diameter.*

Let A B C be the vertical circle, A B the vertical diameter, A C any chord through A, making an angle θ with A B.

Resolve the vertical acceleration g into $g \cos \theta$ along A C, and $g \sin \theta$ perpendicular to A C.

The latter has no effect on the motion of the particle down A C; it therefore starts from rest at A and moves down A C with uniform acceleration $g \cos \theta$.

To find the time it takes, substitute in the formula

$$\begin{aligned}s &= \frac{1}{2} a \cdot t^2, \\A C &= \frac{1}{2} g \cos \theta \cdot t^2; \\\therefore t^2 &= \frac{2 A C}{g \cos \theta}.\end{aligned}$$

Now, since A B is a diameter, A C B is a semicircle, and therefore the angle A C B = 90° ;

$$\therefore \cos \theta = \frac{A C}{A B},$$

or

$$A C = A B \cos \theta.$$

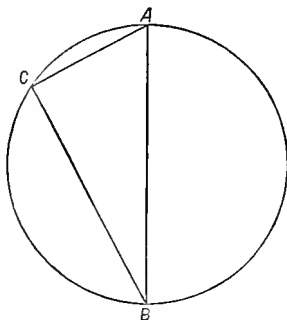


FIG. 122.

Hence,

$$t^2 = \frac{2 AB \cos \theta}{g \cos \theta}$$

$$= \frac{2 AB}{g}.$$

The value of t is thus independent of θ , and is therefore the same for all chords. Further, it is equal to the time down the vertical diameter AB ; for if T were the time taken to fall freely through a vertical height AB we have

$$AB = \frac{1}{2} g \cdot T^2$$

or

$$T^2 = \frac{2 AB}{g}.$$

Similarly we can prove that the times of descent down all chords of a vertical circle which terminate in the *lowest* point of the circle are equal to the time down the vertical diameter.

These results are of considerable importance, since they enable us to find the *line of quickest descent* of a particle in passing from a point to a curve, or *vice versa*.

The *line of quickest descent* is the line along which the time of descent of a particle would be the shortest.

174. *To find the line of quickest descent from a point to a curve in the same vertical plane.*

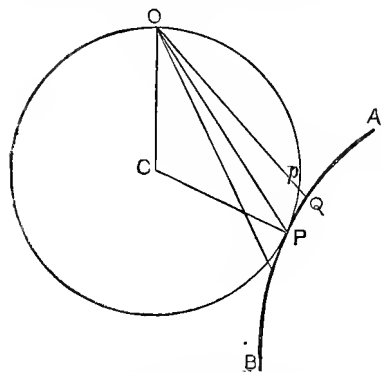


FIG. 123.

Let O be the point, AB the curve.

Describe a circle having its highest point in O and touching the curve in P ; OP is the line of quickest descent.

For draw any other line from O to cut the curve at Q , it must cut the circle before it cuts the curve.

Let it cut the circle in p .

Now, by the last article, the times down OP and $O p$ are equal; $\therefore$ the time down OP is less than the time down OQ ; $\therefore OP$ is the line of quickest descent.

Similarly, to find the line of quickest descent from a curve to a point, we describe a circle touching the curve and having the given point for its lowest point. The line joining the point of contact to the given point is the line of quickest descent.

Note.—The given point may be on the concave or convex side of the curve.

175. In the preceding article, if C be the centre of the circle, CO must be vertical, since O is the highest point. Join CP . Then, since CP is a radius of the circle, it is at right angles to the tangent at P . But the tangent to the circle at P is also the tangent to the curve at P ; therefore CP is normal to the curve at P .

Since $CO = CP$, the angle $COP = CPO$; $\therefore OP$ makes equal angles with the vertical and the normal at P .

Hence the *line of quickest descent* from a point to a curve *bisects the angle between the vertical and the normal to the curve at the point where it meets the curve.*

Similarly the line of quickest descent from a curve to a point bisects the angle between the vertical and the normal at the point where it leaves the curve.

Hence if PQ be the line of quickest descent between two curves, PQ must bisect the angle between the normal to the curve and the vertical at both the points P and Q ; thus the normals, and therefore the tangents, to the two curves at the points P and Q must be parallel.

The chief difficulty in these questions is the geometrical question—How to describe a circle having its centre in a given line to pass through a given point in the line and touch a given curve? The following examples will illustrate this :—

Example 1.—To find the line of quickest descent from a point to an inclined plane.

The line must obviously lie in the vertical plane through the point; hence the question is the same as *to find the line of quickest descent from a point to a line in the same vertical plane.*

Let O be the point and AB the line of greatest slope of the plane in the vertical plane through O .

Describe a circle having O for its highest point and touching the line AB at P .

The centre of this circle may be found as follows :—Since O is the highest point, the circle must touch the horizontal line through O . Let this horizontal line cut AB in B . Now bisect the angle OBA by the line BC , and produce it to cut the vertical through O at the point C . C is the centre of the circle which has O for its highest point and touches AB at P .

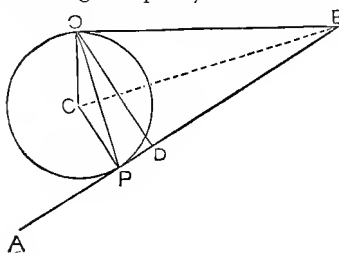


FIG. 124.

Then OP is the line of quickest descent.

The point P might have been found at once, without finding the centre of the circle, by cutting off $BP = BO$, because the tangents drawn from an external point to a circle must be equal.

Otherwise thus :—Draw OD perpendicular to AB , and OC vertically downwards ; bisect the angle COD by the line OP ; OP shall be the line of quickest descent. For, by the last article, OP must make equal angles with the vertical and the normal to the surface at the point P ; but OD is normal to AB and is parallel to the normal at any other point in AB ; hence, since OP bisects

the angle between OD and the vertical line, it will also bisect the angle between the normal and the vertical at P ;

$\therefore OP$ is the line of quickest descent.

Example 2.—To find the line of quickest descent from a given point outside a circle to the circumference.

Let O be the given point, A the centre of the given circle.

Draw a vertical line through O , and cut off OB above the point O equal to the radius of the given circle.

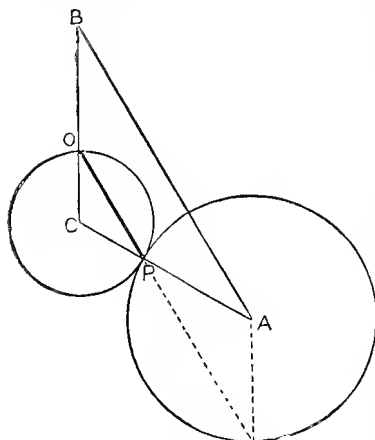


FIG. 125.

Join BA , and make the angle BAC equal to the angle OBA ,

and let AC cut BO produced in C and the given circle in P . C is the centre of the required circle, and OP is the line of quickest descent.

With centre C and radius equal to CO , describe a circle. This touches the given circle at P ;

$\therefore OP$ is the line of quickest descent.

Proof. Because angle $CBA = \text{angle } CAB$;

$\therefore CA = CB$.

But $AP = OB$;

$\therefore CP = CO$;

$\therefore P$ lies on both circles, and since CPA is a straight line, the circles have a common tangent at P , and therefore touch each other at P .

Note.— OP is parallel to AB , and since OB is equal to the radius of this circle, OP produced passes through the lowest point of the circle.

Example 3.—To find the line of quickest descent from a circle to a given point within it.

Let O be the given point, A the centre of the given circle.

Then, as in the previous example, we describe a circle having its *lowest* point at O and touching the given circle at P .

PO is the line of quickest descent.

Again, we note that PO , it produced, passes through the lowest point of the given circle.

Both of the preceding examples might have been solved by means of the following considerations :—

The line of quickest descent must make equal angles with the *vertical* and the normal to the circle at the required point. The normal to the circle at any point is the radius at the point. Any chord of a circle makes equal angles with the radii at its extremities; therefore, the particular chord required must make equal angles with the radius at the required point and the *vertical*

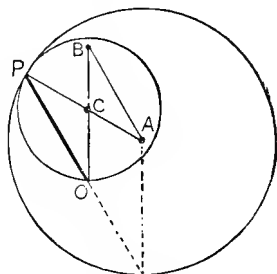


FIG. 126.

radius—that is, it must pass through the extremity of the *vertical* diameter of the given circle.

Example 4.—To find the line of quickest descent between two circles which are external to each other.

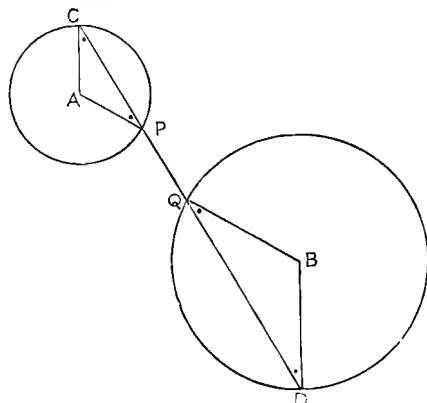


FIG. 127.

Let A and B be the centres of the two circles, C the highest point of the upper one, D the lowest point of the lower one. Join CD, cutting the circumferences again in P and Q.

PQ is the line of quickest descent.

Because $AP = AC$, angle $APC = \text{angle } ACP$; AP is the normal at P, and AC is vertical; $\therefore$ PQ makes equal angle with

the normal at P and the vertical. So also PQ makes equal angle with the normal at Q and the vertical;

$\therefore$ PQ is the line of quickest descent.

To find the time down PQ we must know the length of PQ and the angle it makes with the vertical.

Examples.

1. A body falling from rest describes 402.5 feet in 5 seconds. Find the acceleration. With what velocity must it be projected vertically upwards so as just to reach an elevation of 1,000 feet?

2. A particle is dropped from a height of 50 feet. Find the time which it takes to reach the ground, and the velocity with which it strikes it.

3. A body is dropped from the top of a tower 145 feet high, and strikes the ground with a velocity of 96.6 feet per second. Find the acceleration due to gravity.

4. A ball is dropped from the mast of a ship which is moving

with a velocity of 12 feet per second. Find its actual velocity at the end of half a second.

5. A stone is dropped down a well 144 feet deep ; when it has been falling $1\frac{1}{2}$ second another stone is thrown after it, and the two are heard to strike the bottom together. With what velocity must the second one be projected ?

6. A body which fell from rest described 176 feet in the last second of its fall. Find the total height from which it fell.

7. A body after falling from rest for a certain time describes 112 feet in the next second. What space will it describe in the next 2 seconds ?

8. A stone is let fall from the window of a railway carriage from a point 8 feet above the ground. The train is moving at the rate of 8 miles an hour. Find the angle at which it strikes the ground.

9. A particle is projected vertically upwards with a velocity of 1,000 feet per second. After some time it is observed to pass through 848 feet in two seconds ; how far will it move in the next two seconds, and how long will it then have been moving ?

10. A ball is dropped from a balloon which is ascending at the rate of 10 feet per second, and is 300 feet above the ground. Find the time occupied in the ball's descent.

11. A man, riding in a carriage moving along a level road at the rate of 20 feet per second, throws up a ball and catches it when he is 80 feet farther on the road ; how high does he throw it ?

12. A ball is thrown up a plane inclined to the horizon at an angle of 30° with a velocity of 100 feet per second. Find the time which elapses, and the distance passed over, before the ball begins to return.

13. What velocity will be acquired by sliding from rest down a smooth straight wire 10 feet long, the highest point of which is 6 feet above the lowest point ?

14. In what time will a body slide down 200 feet on a smooth plane inclined at an angle of 30° to the horizontal plane ?

15. A body falls freely from the top of a tower, and during the last second falls four-fifths of the whole distance. Find the height of the tower.

16. The space described by a body in the fifth second of its

fall from rest was to the space described in the last second but three as 9 : 11. For how many seconds did the body fall?

17. A stone, after falling 2 seconds from rest, breaks a pane of glass and consequently loses one-fourth of its velocity. How far will it fall in the next second?

18. An inelastic particle is dropped inside a smooth tube A B C, the upper part of the tube A B is 2 feet long and vertical, the lower part B C $\sqrt{2}$ feet long and inclined at an angle 45° to the horizon. Prove that the velocity of the particle at B is equal to the velocity at C, and show that the whole time down A B C equals $\sqrt{2}$ (the time down A B).

19. A falling body passes two points 10 feet apart in half a second; it subsequently passes two other points also 10 feet apart in one-tenth of a second. Find the distance between the first and last of these four points.

20. A man ascends a tower to a certain height and drops a stone, he then ascends to the top, a distance of 100 feet, and drops another stone; the latter takes half a second longer than the former to reach the ground. Determine the height of the tower.

21. A particle is projected vertically upwards with a velocity of 106 feet per second from a point A, and a second afterwards another particle is projected vertically downwards from a point B above A with a velocity of 30 feet per second. If the two particles reach A and B at the same moment, find the height A B. Find also how long the second particle was falling before it passed the first.

22. Find the inclination of an inclined plane down which a smooth particle slides through a vertical height of 1 foot in half a second.

23. A stone projected vertically downwards with unit velocity from a height of 65 feet strikes the ground four units of time after the instant of the stone's projection. If the unit of space be 3 inches, what is the unit of time?

24. Find the line of quickest descent in the following cases:—

- (1) From a line to a point in the same vertical plane;
- (2) From a point within a circle to the circumference;
- (3) From a circle to a given point outside it.

25. If P Q is the line of quickest descent between two curves,

prove that the vertical lines and the normals to the two curves at the points P and Q form a rhombus.

26. Find the line of quickest descent from a given line to a circle. Determine its length in terms of the distance between the line and the circle, and the angle which the line makes with the horizontal.

27. Find the line of quickest descent between two circles of equal radius.

28. Find the line of quickest descent between two circles of unequal radius, when one circle is inside the other, from the inner circle to the outer circle.

29. Two vertical circles touch each other at their highest point P; through P a chord is drawn, cutting them in A and B. Prove that the time down A B is constant.

30. The distance of a circle from an inclined plane of inclination β is a . Prove that the time down the line of quickest descent from the circle to the plane is $\sqrt{\frac{2a}{g}} \cdot \sec \frac{\beta}{2}$.

31.* If in a circle whose centre is O and highest point H there be drawn two equal chords C P, C Q, subtending an angle α at O, and if the time of falling down C P from rest equals n times that of falling down C Q from rest, and if $\angle H O C = \theta$, prove that

$$\tan \theta = \frac{n^2 - 1}{n^2 + 1} \tan \frac{\alpha}{2}.$$

CHAPTER XIII

PROJECTILES

176. WE shall now consider fully the case of a body projected in any given direction (not vertical), and afterwards acted on by the force of gravity only. Such a body is called a **Projectile**.

We need not now consider the force by which the initial velocity is produced, nor the mass of the moving body; for if mu is the momentum communicated to the particle of mass m in a given direction, the body will continue to move in that direction with uniform velocity u unless acted on by some external force.

If now the force of gravity begins to act upon the body directly it is projected, it communicates to it an acceleration g vertically downwards, so that its path, instead of being a straight line, is now a curve. Here again the value of g is independent of the mass of the body.

Further, by the Second Law of Motion, the change of motion due to any force is the same as if that force alone acted (Art. 34). Consequently the uniform velocity in the direction of projection, and the velocity due to the vertical acceleration g , are not affected by each other, and may be considered independently.

The case has already been considered briefly in Art. 168.

In the following considerations, as in the preceding chapter, we shall assume (1) that the motion of the projectile is *in vacuo*—that is, we do not consider the resistance of the air.

(2) That the direction of the force of gravity is always the same—that is, that the vertical line at any point of the path is parallel to the vertical line at the point of projection.

(3) That the acceleration due to gravity is the same at all points of the path—that is, that every point of the path is equally distant from the earth's centre.

(4) That the effect of the rotation of the earth may be neglected.

(5) That the body is a heavy *particle* (Art. 2).

The errors due to the assumptions (2), (3), and (4) are so very small that they need not be considered in the practical application of our results ; but the modifications depending on the resistance of the air, and the size and shape of the projectile, are so great, that the results of the present chapter are of small practical value.

177. A particle is projected from a given point with initial velocity u in a direction making an angle α with the horizon, and is subject to the action of gravity ; to find its position after any time t .

Let OT be the direction of projection, OY the vertical through O .

The position of the particle at the end of consecutive seconds may be found as in Art. 168. To find the position after t seconds, cut off $OT = ut$, and from OY cut off $OV = \frac{1}{2}gt^2$, and complete the parallelogram $VO TP$, then P is the position of the particle; for if it had had the uniform velocity u only, the particle would have arrived at T , and if it had the acceleration g only, it would have arrived at V ; therefore, by the Second Law of Motion, when it has both these motions it is at P .

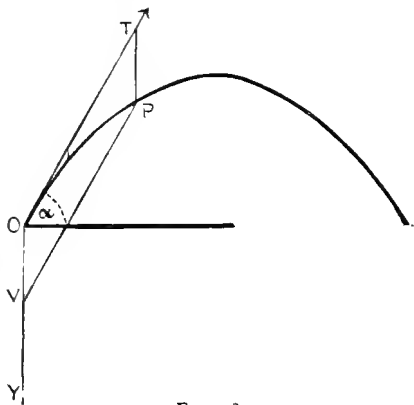


FIG. 123.

If we eliminate t between the equations,

$$O T = u t, \text{ and } O V = \frac{1}{2} g t^2;$$

we have
$$O V = \frac{1}{2} g \left(\frac{O T}{u} \right)^2$$

$$= \frac{g}{2 u^2} \cdot O T^2;$$

$$\therefore O T^2 = \frac{2 u^2}{g} \cdot O V,$$

or
$$P V^2 = \frac{2 u^2}{g} \cdot O V,$$

an equation which shows that the *locus* or path of P is a parabola, since $\frac{2 u^2}{g}$ is constant.

$O T$ is a tangent to the path at the point of projection.

If the student has read Conic Sections, he will at once see that $\frac{u^2}{2 g}$ is the distance of the point O from the directrix, or from the focus of the parabola; for if $O y$ be the *diameter* at the point O , $O T$ the tangent at O , S the focus, P any other point on the curve, and $P V$ be drawn $\parallel$ to $O T$,

$$P V^2 = 4 S O \cdot O V;$$

$$\therefore 4 S O = \frac{2 u^2}{g};$$

$$\therefore S O = \frac{u^2}{2 g}.$$

Hence u , the velocity of projection, is that which would be acquired by a particle falling freely from the directrix to the point of projection.

Definitions.—The path of the particle, which we have proved to be a parabola, is sometimes called its *trajectory*.

The angle α which the direction in which it is projected makes with the horizon is called the **angle of projection**.

The distance between the point of projection and the point where the path of the particle again meets the horizontal plane through the point of projection is called the **horizontal range**,

and the time that the particle takes over this portion of its path is called the **time of flight**.

178. *A particle is projected into space with an initial velocity u in a direction making an angle α with the horizon. To find the greatest height which the particle attains, the time of flight, and the horizontal range.*

The velocity of projection u may be resolved into its two components— $u \cos \alpha$ horizontally, and $u \sin \alpha$ vertically upwards. Of these $u \cos \alpha$ remains constant, but $u \sin \alpha$ is subject to the vertical retardation g . The distances traversed by the particle on account of these two motions may be considered independently of each other.

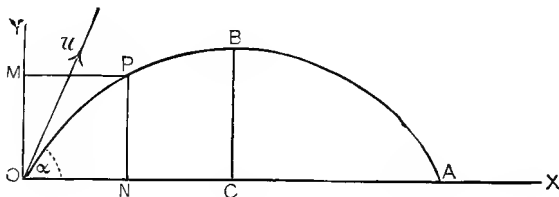


FIG. 129.

Thus, let P be the position of the particle after any time t , let O x , O y be the horizontal and vertical lines through O in the plane of projection. Draw P N, P M perpendicular to O x , O y . Then

O N = distance due to the horizontal velocity = $u \cos \alpha \cdot t$;
 P N = O M = distance due to the vertical velocity and retardation $g = u \sin \alpha \cdot t - \frac{1}{2} g t^2$.

The particle will attain its greatest height above the horizontal plane O A when the vertical velocity is reduced to zero by the retardation g .

Hence, to find the greatest height B C ($= z$) and the time from O to B ($= t'$), we put $v = 0$ in the equations connecting the vertical velocity with the vertical distance and the time.

$$v^2 = (u \sin \alpha)^2 - 2 g s,$$

$$0 = u^2 \sin^2 \alpha - 2 g z; \quad \therefore z = \frac{(u \sin \alpha)^2}{2 g}$$

$$v = u \sin \alpha - g t,$$

$$0 = u \sin \alpha - g t'; \quad \therefore t' = \frac{u \sin \alpha}{g}.$$

At the point B the velocity of the particle is entirely horizontal and $= u \cos \alpha$.

The particle now descends, and the time that it takes to reach the horizontal plane again must be equal to the time it took to get from O to B (Art. 171).

Hence, if T be the *time of flight*,

$$T = 2 t' = \frac{2 u \sin \alpha}{g}.$$

Since the horizontal velocity is uniform, the horizontal range $=$ horizontal velocity $\times$ time of flight. Hence, if R be the *horizontal range*,

$$\begin{aligned} R &= u \cos \alpha \times T = u \cos \alpha \times \frac{2 u \sin \alpha}{g} \\ &= \frac{2 u^2 \sin \alpha \cos \alpha}{g}. \end{aligned}$$

By trigonometry $2 \sin \alpha \cos \alpha = \sin 2 \alpha$;

$$\therefore R = \frac{u^2 \sin 2 \alpha}{g}.$$

The time of flight might have been found at once, without first finding the time from O to B, by putting $P N = 0$ in the equation connecting the vertical distance with the time,

$$P N = u \sin \alpha \cdot t - \frac{1}{2} g t^2.$$

Put $P N = 0$,

$$t \left(\frac{1}{2} g t - u \sin \alpha \right) = 0;$$

$$\therefore t = 0, \text{ or } t = \frac{2 u \sin \alpha}{g}.$$

The first value denotes that $P N = 0$ when the particle starts; the second value of t is the time of flight, since $P N$ is again zero when the particle is at A.

Note.—The particle will have acquired the same vertical velocity in its descent at the point A as it had on its ascent when it started at O, and since the horizontal velocity is always the same, the actual velocity at A is the same as at O, and makes the same angle with the horizon. Similarly, it can be proved that the velocities of the particle at any two points equally distant from the horizon are equal to each other and are equally inclined to the horizon.

179. *To find the angle of projection when the horizontal range is a maximum for a given velocity of projection.*

$$R = \frac{2u^2 \sin \alpha \cos \alpha}{g}.$$

R is a maximum when $\sin \alpha \cos \alpha$ is a maximum.

Now $2 \sin \alpha \cos \alpha = 1 - (\sin \alpha - \cos \alpha)^2$.

Hence $2 \sin \alpha \cos \alpha$ is greatest when $(\sin \alpha - \cos \alpha)^2$ is least,

when $\sin \alpha - \cos \alpha = 0$,

when $\sin \alpha = \cos \alpha$,

when $\alpha = 45^\circ$;

$\therefore$ the range is a maximum when $\alpha = 45^\circ$,

in which case $R = \frac{u^2}{g}$.

Hence, **maximum range** for a given velocity of projection

$$= \frac{u^2}{g}.$$

Or thus:

$$R = \frac{u^2 \sin 2\alpha}{g}.$$

R is a maximum when $\sin 2\alpha$ is a maximum,

i.e. when $\sin 2\alpha = 1$,

when $2\alpha = 90^\circ$,

when $\alpha = 45^\circ$;

then $R = \frac{u^2}{g}$.

It will be noticed that the value of R for given values of u and α remains the same if we substitute $90^\circ - \alpha$ for α ; hence there are two directions of projection, so that a body projected with a given velocity should have a certain range.

These two directions are equally inclined to the horizon and vertical respectively; hence they make equal angles with the direction of projection when the range is a maximum.

180. *To find the velocity and direction of the particle after any time t .*

Let P be the position of the particle.

Let v be the velocity, θ the angle the direction of the particle makes with the horizon when it is at the point P .

Then $v \cos \theta =$ horizontal velocity at P ,

$v \sin \theta =$ vertical velocity at P .

$$v \cos \theta = u \cos \alpha \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$v \sin \theta = u \sin \alpha - g t \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \tan \theta = \frac{u \sin \alpha - g t}{u \cos \alpha},$$

from which we can find θ .

To find v , square (1) and (2) and add them; then

$$(v \sin \theta)^2 + (v \cos \theta)^2 = (u \sin \alpha - g t)^2 + (u \cos \alpha)^2;$$

$$\text{or } v^2 (\sin^2 \theta + \cos^2 \theta) = u^2 (\sin^2 \alpha + \cos^2 \alpha) - 2 u \sin \alpha \cdot g t + g^2 t^2;$$

$$\therefore v^2 = u^2 - 2 g (u \sin \alpha \cdot t - \frac{1}{2} g t^2).$$

But if $y =$ vertical distance of P above the point of projection,

$$y = u \sin \alpha \cdot t - \frac{1}{2} g t^2;$$

$$\therefore v^2 = u^2 - 2 g y.$$

Range on an Inclined Plane.

181. A particle is projected with velocity u at an angle α to the horizon from a point on a plane of inclination β , the direction of projection being in the vertical plane perpendicular to the inclined plane. Find the range on the inclined plane.

Let OA be the range on the plane, Ox horizontal. Draw AN perpendicular to Ox .

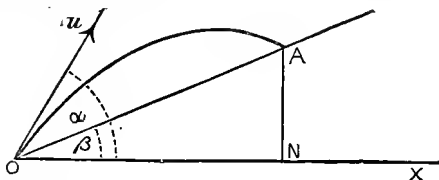


FIG. 130.

The velocity u may be resolved into its components

$u \cos (\alpha - \beta)$ along the plane,

and $u \sin (\alpha - \beta)$ perpendicular to the plane.

The retardation g may be resolved into its components

$g \sin \beta$ down the plane,

and $g \cos \beta$ perpendicular to the plane.

To find T , the time from O to A , put $s = 0$ in the equation which connects the distance of the particle *perpendicular* to the plane with the time,

$$s = u \sin (\alpha - \beta) t - \frac{1}{2} g \cos \beta \cdot t^2.$$

Put $s = 0$;

$$\therefore t = 0, \text{ or } \frac{2 u \sin (\alpha - \beta)}{g \cos \beta}.$$

Hence
$$T = \frac{2 u \sin (\alpha - \beta)}{g \cos \beta}.$$

Now, ON = distance travelled in time T on account of the horizontal velocity $u \cos \alpha$ which is uniform ;

$$\therefore ON = u \cos \alpha \cdot T = \frac{2 u^2 \cos \alpha \cdot \sin (\alpha - \beta)}{g \cos \beta}.$$

But
$$\frac{ON}{OA} = \cos \beta ;$$

$$\therefore OA = \frac{ON}{\cos \beta} ;$$

$$\therefore \text{Range} = \frac{2 u^2 \cos \alpha \cdot \sin (\alpha - \beta)}{g \cos^2 \beta}.$$

If the plane were turned round so that it inclined downwards from the point of projection instead of upwards, the range down the plane would be given by

$$R = \frac{2 u^2 \cos \alpha \sin (\alpha + \beta)}{g \cos^2 \beta}.$$

This result may be worked out independently, or it may be obtained by putting $-\beta$ for β in the former result.

It may also be obtained from the former result by supposing that the direction of projection is turned round so as to make an angle α with the horizon on the other side of the point of projection; we therefore substitute $180^\circ - \alpha$ for α , and get

$$\begin{aligned} R &= \frac{2 u^2 \cos (180^\circ - \alpha) \sin (180^\circ - \alpha - \beta)}{g \cos^2 \beta} \\ &= -\frac{2 u^2 \cos \alpha \sin (\alpha + \beta)}{g \cos^2 \beta}. \end{aligned}$$

The negative sign means that the range is then down the plane instead of up the plane.

182. *Maximum range on an inclined plane.*

$$\begin{aligned} R &= \frac{2 u^2 \cos \alpha \sin (\alpha - \beta)}{g \cos^2 \beta} \\ &= \frac{u^2}{g \cos^2 \beta} \{2 \cos \alpha \sin (\alpha - \beta)\} \\ &= \frac{u^2}{g \cos^2 \beta} \{\sin (2 \alpha - \beta) - \sin \beta\}. \end{aligned}$$

Since u , g , and β are constant,

R is a maximum when $\sin (2 \alpha - \beta)$ is a maximum,

when $\sin (2 \alpha - \beta) = 1$,

when $2 \alpha - \beta = 90^\circ$,

when $\alpha = 45^\circ + \frac{\beta}{2}$;

that is, when the direction of projection bisects the angle between the plane and the vertical, in which case

$$\begin{aligned} R &= \frac{u^2}{g \cos^2 \beta} (1 - \sin \beta) \\ &= \frac{u^2}{g (1 + \sin \beta)} = \frac{u^2}{2 g \sin^2 \alpha'} \end{aligned}$$

Here also it will be noticed that the general value of R at the beginning of the article will be the same for the direction of projection α' as it is for α ,

provided $\sin (2\alpha' - \beta) = \sin (2\alpha - \beta)$;
 that is, if $2\alpha' - \beta = 180^\circ - (2\alpha - \beta)$,
 if $\alpha' = 90^\circ - (\alpha - \beta)$.

Now the direction of projection for the maximum range is $45^\circ + \frac{\beta}{2}$;

$$\begin{aligned} \therefore \alpha' - \left(45^\circ + \frac{\beta}{2}\right) &= 90^\circ - \alpha + \beta - 45^\circ - \frac{\beta}{2} \\ &= 45^\circ - \alpha + \frac{\beta}{2} \\ &= \left(45^\circ + \frac{\beta}{2}\right) - \alpha \end{aligned}$$

Hence the two directions of projection for a given range make equal angles with the direction for the maximum range.

If there is only one direction of projection for a given range, that range is the maximum range for the given velocity of projection.

183. The particle is at its greatest distance (z) from the plane when the velocity perpendicular to the plane is entirely destroyed by the component of g perpendicular to the plane.

Putting $v = 0$ in the equation

$$v^2 = \{u \sin (\alpha - \beta)\}^2 - 2 g \cos \beta \cdot s,$$

we have

$$z = \frac{u^2 \sin (\alpha - \beta)^2}{2 g \cos \beta}.$$

The condition that the particle should strike the plane at right angles is found by expressing the fact that the velocity *parallel* to the plane is zero at the end of the time of flight.

Putting $v = 0$ when $t = T$ in the equation

$$v = u \cos (\alpha - \beta) - g \sin \beta \cdot t,$$

we have $0 = u \cos (\alpha - \beta) - g \sin \beta \cdot T$;

$$\therefore u \cos (\alpha - \beta) = g \sin \beta \cdot \frac{2 u \sin (\alpha - \beta)}{g \cos \beta},$$

or
$$\frac{\cos (\alpha - \beta)}{\sin (\alpha - \beta)} = \frac{2 \sin \beta}{\cos \beta},$$

or
$$1 = 2 \tan \beta \cdot \tan (\alpha - \beta);$$

whence
$$\tan \alpha = \frac{1 + 2 \tan^2 \beta}{\tan \beta}.$$

184. To find the elements of the parabolic path.

Let OT be the direction of projection, and hence the tangent to the path at the point O.

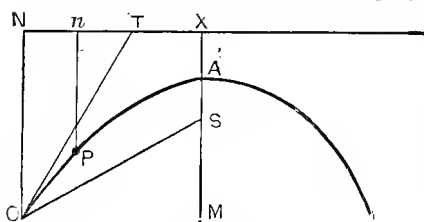


FIG. 131.

Let A be the highest point, and draw AM vertical. A is the vertex of the parabola, and AM is the axis.

We have shown in Article 177 that if u

be the velocity of projection,

$$u^2 = 2g \cdot SO = 2g \cdot ON,$$

where ON is the perpendicular from O on the directrix.

Hence the velocity of projection is equal to the velocity which would be acquired by a particle falling freely through the height ON—that is, through the height from the directrix to the point of projection.

Now we may regard any point of the path as the point of projection, and suppose that the particle is projected from that

point along the tangent to the parabola at that point with the velocity which it then has.

Thus the velocity at P = velocity due to the height P n; if v = velocity at P, $v^2 = 2g \cdot P n$.

Similarly at A, the highest point,

vel. at A = velocity due to the height A X.

But the vel. at A = $u \cos \alpha$;

$$\therefore (u \cos \alpha)^2 = 2g \cdot AX;$$

$$\therefore AX = \frac{u^2 \cos^2 \alpha}{2g}.$$

But the latus rectum of a parabola = $4 AX$;

$$\therefore \text{the latus rectum} = \frac{2 u^2 \cos^2 \alpha}{g}.$$

Alternative proof.—We have already found that A M, the greatest height of the projectile = $\frac{u^2 \sin^2 \alpha}{2g}$, and we have also proved that O N, the distance of O from the directrix = $\frac{u^2}{2g}$;

$$\begin{aligned} \therefore AX &= ON - AM, \\ &= \frac{u^2}{2g} - \frac{u^2 \sin^2 \alpha}{2g} \\ &= \frac{u^2}{2g} (1 - \sin^2 \alpha) \\ &= \frac{u^2 \cos^2 \alpha}{2g}. \end{aligned}$$

The value of the latus rectum, or of AS, being known in terms of the velocity and angle of projection, the size of the parabola is completely determined.

Note.—The latus rectum of the parabola depends on $u \cos \alpha$ only; hence it is independent of the vertical velocity. Thus all particles which have the same horizontal velocity will describe the same sized path, whatever be their vertical velocity.

185. A great deal of the work in this subject admits of

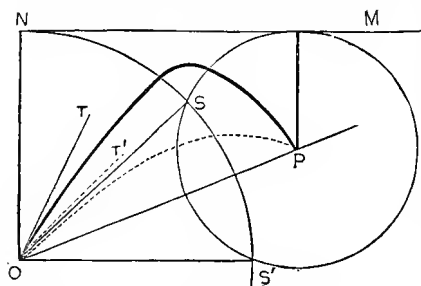


FIG. 132.

an easy geometrical solution if the student is familiar with the geometry of the parabola.

We will take, as an example, the consideration of the two directions of projection for a given range on an inclined plane,

the direction of projection for the maximum range, and the maximum range.

Let O be the point of projection, O P the required range.

Draw O N vertically upwards equal to $\frac{u^2}{2g}$; then N is a point on the directrix, and N M, perpendicular to N O, is the directrix. The first part of the problem now resolves itself into finding the parabola which passes through the points O and P and whose directrix is N M.

Draw P M $\perp$ to N M. With centres O and P, and radii O N and P M respectively, describe circles cutting each other in S and S'; then, since S O = S' O = O N, and S P = S' P = P M, S and S' are the foci of the two paths which have the required range O P. But the direction of projection is the tangent to the parabola at the point of projection, and the tangent at any point of a parabola bisects the angle between the focal distance and the line drawn perpendicular to the directrix from the point; hence the required directions of projection are O T and O T', which bisect the angles S O N, S' O N respectively.

It is easy to show that angle T O N = angle T' O P.

$$\begin{aligned}
 \text{For } \quad \text{T O N} &= \frac{1}{2} \text{ S O N} \\
 &= \frac{1}{2} (\text{S' O N} - \text{S' O S}) \\
 &= \text{S' O T'} - \text{S' O P} \\
 &= \text{P O T'};
 \end{aligned}$$

Then

$$\begin{aligned}
 R &= OS + SP \\
 &= ON + PM \\
 &= ON + ON - PK \\
 &= 2 ON - R \sin \beta;
 \end{aligned}$$

$$\therefore R(1 + \sin \beta) = 2 ON,$$

or

$$R = \frac{2 ON}{1 + \sin \beta} = \frac{u^2}{g(1 + \sin \beta)}.$$

To find the velocity at P when the range is a maximum.

$$\begin{aligned}
 v^2 &= 2g \cdot PM = 2g \cdot PS = 2g(R - ON) \\
 &= 2g \cdot ON \left(\frac{2}{1 + \sin \beta} - 1 \right) = u^2 \cdot \frac{1 - \sin \beta}{1 + \sin \beta} \\
 &= u^2 \cdot \frac{(1 - \sin \beta)^2}{\cos^2 \beta} \\
 \therefore v &= u \cdot \frac{1 - \sin \beta}{\cos \beta} = u(\sec \beta - \tan \beta).
 \end{aligned}$$

186.* Sometimes in solving problems we shall find it advantageous to use the *equation* to the path referred to the horizontal and vertical lines through the point of projection as axes.

Definition.—The equation to a curve referred to any two lines as rectangular axes expresses the relation which must exist between the distances of any point on the curve from these two lines.

These distances are called the co-ordinates of the point. (See Trigonometry, Art. 4.)

Thus, in Article 178, if Ox , Oy are the rectangular axes through the point of projection x , y the co-ordinates of P after any time t ,

$$\begin{aligned}
 x &= ON = u \cos \alpha \cdot t, \\
 y &= PN = u \sin \alpha \cdot t - \frac{1}{2} g t^2.
 \end{aligned}$$

Eliminate t .

$$y = u \sin \alpha \left(\frac{x}{u \cos \alpha} \right) - \frac{1}{2} g \left(\frac{x}{u \cos \alpha} \right)^2,$$

$$y = x \tan \alpha - \frac{g x^2}{2 u^2 \cos^2 \alpha}.$$

The curve represented by this equation must be a parabola whose latus rectum is $\frac{2 u^2 \cos^2 \alpha}{g}$.

Thus, to find the range on an inclined plane, let β be the inclination of the plane and R = the range ; then

$$x = R \cos \beta, \text{ and } y = R \sin \beta ;$$

$$\therefore R \sin \beta = R \cos \beta \cdot \tan \alpha - \frac{g R^2 \cos^2 \beta}{2 u^2 \cos^2 \alpha} ;$$

$$\therefore g R^2 \cos^2 \beta = R (\cos \beta \tan \alpha - \sin \beta) \times 2 u^2 \cos^2 \alpha ;$$

$$\begin{aligned} \therefore R &= \frac{2 u^2 \cos^2 \alpha}{g \cos^2 \beta} \left(\frac{\sin (\alpha - \beta)}{\cos \alpha} \right) \\ &= \frac{2 u^2}{g} \cdot \frac{\cos \alpha \sin (\alpha - \beta)}{\cos^2 \beta}. \end{aligned}$$

187. Many experiments have been made with the object of determining a practical formula for the horizontal range of a projectile when the resistance of the air and the size and weight of the projectile were taken into account.

Up to a certain point the resistance of the air increases with the velocity. A large number of experiments carried on in 1861 led Professor Hélie to the conclusion that the resistance of the air at practical velocities was more nearly proportional to the cube of the velocity than to any other working expression.

He constructed a formula for the trajectory by empirically modifying the formula given above for the path of the projectile *in vacuo* thus—

$$y = x \tan \alpha - \frac{g x^2}{2 \cos^2 \alpha} \left(\frac{1}{u^2} + \frac{k x}{u} \right),$$

where

$$k = 0.0000000458 \frac{d^2}{w},$$

d = diameter of projectile in inches,

w = weight of projectile in pounds.

The above value of k is for ogival heads which suffer least retardation from the air ; with other shaped heads a different coefficient of resistance would be required.

Putting $y = 0$ in the above equation, and dividing the equation by x and simplifying, we get the following quadratic equation for x the horizontal range—

$$gx (1 + k u x) = u^2 \sin 2a = R \cdot g,$$

where R is the range of a particle *in vacuo*, which gives

$$x = \frac{1}{2ku} \{ \sqrt{1 + 4ku \cdot R} - 1 \},$$

an expression for the range which at low elevations for initial velocities between 800 f.s. and 1,400 f.s. is very fairly accurate, and which furnishes the simplest method yet devised of roughly constructing a range-table. (Encyc. Brit. 9th Edit. vol. xi. p. 302.)

If we expand the surd in the above expression by the binomial theorem, and neglect terms which contain higher powers of k than the second, we get

$$\begin{aligned} x &= \frac{1}{2ku} \left\{ 1 + \frac{1}{2} (4kuR) - \frac{1}{8} (4kuR)^2 - 1 \right\} \\ &= \frac{1}{2ku} \{ 2kuR - 2k^2 u^2 R^2 \} \\ &= R \{ 1 - k u R \}. \end{aligned}$$

Example 1.—A bullet is projected with a velocity of 200 feet per second at an angle of 30° with the horizon. Find the horizontal range and the time of flight; find also the maximum range for this velocity of projection.

$$\text{Horizontal velocity} = 200 \cos 30^\circ = 100 \sqrt{3}.$$

$$\text{Vertical velocity} = 200 \sin 30^\circ = 100$$

$$\text{Time of flight} = \frac{200}{g} \text{ seconds.}$$

$$\begin{aligned} \text{Range} &= 100 \sqrt{3} \times \frac{200}{g} \\ &= \frac{20000 \sqrt{3}}{g} = 1,020 \text{ feet.} \end{aligned}$$

$$\text{Maximum range} = \frac{u^2}{g} = \frac{40000}{g} = 1,250 \text{ feet.}$$

Example 2.—A particle is projected from the top of a perpendicular cliff h feet high with a horizontal velocity u . Find the distance from the cliff of the point when it strikes the ground.

The time it is in the air = time it would take to fall freely through h feet. If t be the time, $h = \frac{1}{2} g t^2$,

$$\therefore t = \sqrt{\frac{2h}{g}}.$$

The horizontal velocity u is constant ;

$\therefore$ the distance of the particle from the cliff when it strikes the ground

$$\begin{aligned} &= u t = u \sqrt{\frac{2h}{g}} \\ &= u \sqrt{\frac{2h}{32}} \\ &= \frac{u\sqrt{h}}{4} \text{ feet.} \end{aligned}$$

Example 3.—A man can just throw a cricket ball 108 yards. With what velocity does the ball leave his hand, how high will it rise, and how long will it be in the air ?

If u be the initial velocity, the maximum range $= \frac{u^2}{g}$;

$$\frac{u^2}{g} = 324 ;$$

$$\therefore u^2 = 324 g,$$

$$u = 72 \sqrt{2}$$

$$= 102 \text{ feet per second nearly.}$$

The direction of projection for maximum range $= 45^\circ$;

$\therefore$ vertical velocity $= 72$ feet per second $=$ horizontal velocity.

$$\text{Greatest height} = \frac{(72)^2}{2g} = 81 \text{ feet.}$$

$$\text{Time of flight} = \frac{2 \times 72}{g} = 4\frac{1}{2} \text{ seconds.}$$

Example 4.—A number of particles are projected at the same instant with different velocities from the same point in the same direction and in the same vertical plane. Prove that at any subsequent time they all lie in a straight line.

Let O A B be the common direction of projection, and suppose that A and B would be the positions of any two of the particles after a time t if gravity did not act ; now, in consequence of the action of gravity, they are at A' and B' vertically beneath A and B, such that $AA' = BB' = \frac{1}{2}gt^2$; $\therefore$ A' B' is parallel to A B ; hence after any time all the particles lie on a line parallel to the direction of projection.

Example 5.—A particle is projected from a point O with velocity u at an angle α ; P is its position at any subsequent time. If A be the highest point, PN the perpendicular from P on the vertical through A, prove (by dynamical considerations) that

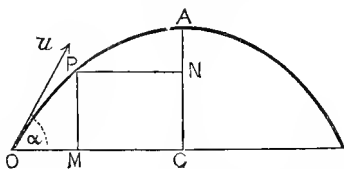


FIG. 134.

$$PN^2 = AN \times \text{constant.}$$

Draw $PM \perp$ to OC . Then

$$AC = \frac{u^2 \sin^2 \alpha}{2g},$$

and $OC = \frac{u \sin \alpha}{g} \cdot u \cos \alpha;$

$$PN = OC - OM$$

$$= \frac{u^2 \sin \alpha \cos \alpha}{g} - u \cos \alpha \cdot t$$

$$= \frac{u \cos \alpha}{g} (u \sin \alpha - gt);$$

$$AN = AC - NC$$

$$= AC - PM$$

$$= \frac{u^2 \sin^2 \alpha}{2g} - (u \sin \alpha \cdot t - \frac{1}{2}gt^2);$$

$$\therefore PN^2 = \frac{u^2 \cos^2 \alpha}{g^2} (u \sin \alpha - gt)^2$$

$$= \frac{u^2 \cos^2 \alpha}{g^2} (u^2 \sin^2 \alpha - 2u \sin \alpha \cdot gt + g^2 t^2)$$

$$= \frac{2u^2 \cos^2 \alpha}{g} \left(\frac{u^2 \sin^2 \alpha}{2g} - u \sin \alpha \cdot t + \frac{1}{2}gt^2 \right)$$

$$= \frac{2u^2 \cos^2 \alpha}{g} \cdot AN;$$

$$\therefore PN^2 = \text{constant} \times AN.$$

This result proves that P is on a parabola whose axis is AC, and whose latus rectum is $\frac{2u^2 \cos^2 \alpha}{g}$.

Example 6.—An army is encamped on a hill h feet high. An opposing force is on the level ground at such a distance from the hill that it subtends an angle β . What is the least velocity of

projection that will enable a cannon ball to reach the top of the hill?

The distance of the top of the hill from the opposing force is $\frac{h}{\sin \beta}$. This must be the maximum range on a plane whose inclination is β for the least velocity of projection.

Hence if u be the least velocity of projection,

$$\frac{u^2}{g(1 + \sin \beta)} = \frac{h}{\sin \beta} \quad (\text{Art. 182.})$$

$$\begin{aligned} u^2 &= \frac{g h (1 + \sin \beta)}{\sin \beta} \\ &= g h (\operatorname{cosec} \beta + 1). \end{aligned}$$

Examples.

1. If the greatest height attained by a projectile be 36 feet, prove that the vertical velocity is 48.

2. A bullet is projected at an angle $\cos^{-1} \frac{4}{5}$ to the horizon with a velocity of 1,200. Find the greatest height and the horizontal range.

3. If the velocity of projection be 660 feet per second, find the maximum range in miles.

4. What is the direction of projection when the horizontal range is four times the greatest height?

5. Determine the direction of projection when the greatest height acquired is equal to the horizontal range.

6. If the horizontal range of a cannon ball be 12 miles when the direction of projection is $\sin^{-1} \frac{3}{5}$, find the velocity at the highest point of the path.

7. A stone is projected at an angle 30° with the horizon. Find the velocity of projection in order that the greatest height may be 9 feet above the horizontal plane through the point of projection, and find the range on this plane.

8. A bullet leaves a gun with a horizontal velocity of 1,200 feet per second and a vertical velocity of 20 feet per second. Find the greatest height and the horizontal range.

9. If V be the velocity and α the angle of projection, and if v be the velocity and β the angle which the direction of v makes with the horizontal at any subsequent time, $v = V \cos \alpha \cdot \sec \beta$.

10. A body is projected with a velocity of 160 feet per second at an angle α with the horizon, determine the velocity with which it strikes an object 100 feet above the ground.

11. If H be the greatest height of a projectile, R the horizontal range, V the velocity of projection, α the angle of projection, prove that $\tan \alpha = \frac{4H}{R}$, and that $R^2 = 16H \left(\frac{V^2}{2g} - H \right)$.

12. Prove that the maximum horizontal range of a projectile is equal to the latus rectum of the parabolic path.

13. Prove that the maximum range is equal to twice the height to which the velocity of projection is due.

14. A man can just throw a stone vertically upwards to a height of 100 feet; how far could he send it in the same horizontal plane?

15. The maximum range of a projectile is 1,600 yards. Find the time of flight.

16. A stone is thrown at an angle of elevation of 60° , so as just to strike the top of an object 54 feet high, whose base is on the same level as the point of projection, and which subtends an angle of 30° at that point. Find what the velocity of projection must be, and determine the time of flight.

17. If a particle be projected with velocity u at an angle α from a point in an inclined plane of inclination β , then its inclination to the horizon at the point where it strikes the plane is given by the equation $\tan \theta = 2 \tan \beta - \tan \alpha$.

18*. If a projectile strike an inclined plane (β) at right angles, the range on the plane is

$$\frac{2u^2}{g} \cdot \frac{\sin \beta}{1 + 3 \sin^2 \beta}$$

19. A smooth square board, whose side is 51 feet, is placed with two of its sides horizontal and is inclined at an angle 45° to the horizon. With what velocity must a particle be projected horizontally in the plane of the square from one of the top corners in order that it may strike the opposite corner? (Take $\sqrt{2} = 1\frac{5}{12}$.)

20. A body is projected from a given point with velocity v at an angle α to the horizon. When it reaches the highest point another body is projected from the same point with velocity v' at an angle α' so as to strike it. Prove that

$$\frac{v}{v'} = \frac{\cos \alpha'}{\cos \alpha} - \frac{2 \sin \alpha'}{\sin \alpha}$$

21. If at two instants during the motion of a projectile the direction of motion make angles α, β with the horizontal, then the time between the two observations is $(\tan \alpha - \tan \beta) \frac{u}{g}$, where u is the horizontal component of the velocity of the projectile.

22.* A shell is projected with velocity V at an angle α with the horizon, and when at its highest point it explodes into two equal parts, one of which rises vertically to a height equal to the height at the moment of explosion ; where will the other part strike the ground ?

23. If two particles be projected from the same point in the same vertical plane with equal velocities (v) in different directions, so as to have the same horizontal range (R), find the difference between their times of flight.

24. If the velocity of projection be that due to the vertical height a , and h be the greatest height reached by the projectile, then the horizontal range $= 4 \sqrt{a h - h^2}$.

25. Prove that the expression for the range in the preceding question is a maximum when $h = \frac{1}{2} a$.

26.* A particle, after sliding down an inclined plane, moves in a parabola of given latus rectum. Find the inclination and length of the plane in order that the focus of the parabola may be vertically below the point from which the particle started.

27. Prove that the velocity at any point of the path of a projectile varies as the perpendicular let fall from the focus on its direction at that point.

28. Prove that the line joining the two foci of the paths which have the same range on an inclined plane is perpendicular to the plane.

29.* Two particles are projected with the same velocity from the same point, but with different inclinations, in the same plane ; prove that when their directions are parallel they are moving at right angles to the line which bisects the angle between their directions of projection.

30. If α be the direction in order that a particle with a given velocity of projection should pass through a given point, find an equation for $\tan \alpha$. What is the condition that the roots of this equation should be equal ? Hence find an expression for the maximum range.

31.* A particle is projected at an elevation of 45° from the extremity of the horizontal diameter of a vertical circle with such a velocity that, if it had been projected vertically upwards, it would have risen to a height of one-eighth of the vertical diameter. Find where it will strike the circle.

32.* The maximum horizontal range for a particle with a given velocity being R , determine the elevation necessary for it to strike an object distant $2 R \sec \alpha (\tan \alpha)^{\frac{1}{2}}$ on a plane through the point of projection depressed α° below the horizon.

CHAPTER XIV

FURTHER CONSIDERATIONS OF THE LAWS OF MOTION

188. THE three Laws of Motion as given by Newton are as follows :—

LEX I.—*Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum suum mutare.*

Every body continues in its state of rest or of uniform motion in a straight line except in so far as it is compelled by impressed forces to change that state.

LEX II.—*Mutationem motus proportionalem esse vi motrici impressæ et fieri secundum lineam rectam qua vis illa imprimitur.*

Change of motion is proportional to the impressed force, and takes place in the straight line in which that force is impressed.

LEX III.—*Actioni contrariam semper et æqualem esse reactionem : sive, corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.*

To every action there is always an equal and opposite reaction ; that is to say, the mutual actions of any two bodies are always equal and in opposite directions.

There is no possible *à priori* proof of these laws. They have been deduced in the first instance from observation and experiment, and from them the whole science of dynamics and astronomy has been built up ; if rigorous proof be looked for, it will be found in the fact that the ‘ Nautical Almanac ’ for each year is published some years in advance, and that the

predictions contained in it concerning the places of the heavenly bodies from day to day are found to agree most exactly with our observations. These predictions depend on the theory of astronomy, which is based on the laws of motion ; we cannot, therefore, but believe in the truth of these laws, since the results which are deduced from them are so universally corroborated. Any deviation between deduction and observation would not make us doubt the truth of our laws, but would lead us to the discovery of the disturbing cause ; and, in fact, such deviations in the past have led to the discovery of a planet whose existence was not previously known.

189. Though called Newton's laws of motion, it cannot be said that they were discovered by him. Previous to the time of Galileo the popular ideas about force and motion were in a very confused state ; Galileo was the first to put Laws 1 and 2 into a satisfactory form, and it is probable that his death (1641) alone prevented him from arriving at a satisfactory statement of Law 3. About 1669 papers were sent to the Royal Society of London by Wren, Wallis, and Huyghens, which show that people were gradually coming to apprehend the third law of motion in its most general sense. These papers stated that in the mutual impact of two bodies 'the quantity of motion remains unaltered,' or 'the momentum gained by the one body is equal to the momentum lost by the other.'

Newton was the first to collect the knowledge acquired by his predecessors—much of which he extended and greatly improved upon—and to present the whole subject in a systematic and intelligible treatise. (Whewell's 'History of the Inductive Sciences,' Book VI.)

190. As we have already seen in Chapter I., the first law states the property of matter called Inertia, by virtue of which it maintains its state of rest or uniform motion in a straight line ; it also gives us our definition of Force, as that which changes the inertia of a body. If a body changes its state of inertia, it must be on account of the action of some force or forces ; if a body's state remain the same, no force acts upon it. Newton extends this property also to the state of a body which is rotating about an axis, as in the case of a hoop ; so a

wheel rotating about an axis through its centre will maintain this state of rotation if it is not acted upon by forces such as the resistance of the air and the friction of the axis in the sockets.

In no case can we prove this law by experiment, since it is not possible in nature to entirely eliminate the forces which resist motion ; but the more nearly we eliminate these forces, the more nearly does the motion of a body, unacted on by force, approach to being uniform.

The nearest approach to uniform motion which we can observe in nature is the rotation of the earth on its axis. This motion is scarcely altered at all by external forces, and is so nearly uniform that we use it as our measurement of time, and say that *equal intervals of time are the times during which the earth turns through equal angles.*

Thus, if 24 hours be the time which the earth takes to rotate through an angle of 360° , it rotates through 15° per hour, or $15'$ per minute, or $15''$ per second.

191. The second law not only contains the first law by implication, but makes the definition of force, which is contained in it, more complete by supplying us with a measure of force.

The word *motus* in the original stands for *quantitas motus*, or quantity of motion, to which we always give the name momentum. The second law says 'that change of momentum is proportional to the force which produces it,' implying that if there is no change of momentum there is no force, and *vice versa* ; and further adds that if there is a change of momentum in any direction it is due entirely to the force which acts in that direction, which force is proportional to, and may therefore be measured by, that change of momentum. We therefore choose our unit force to be the force which produces in one second a change in momentum of one unit of momentum, and we measure a force of any magnitude by the change of momentum which it produces in one second in the direction in which it acts. Thus, $F = ma$; where F is the force, m the mass of the body it acts upon, a the acceleration produced

in the direction of the force. (See Chap. I., and especially Art. 26.)

Further, the second law implies that the change of momentum due to any force which acts on a body is the same whether that body be at rest or in motion, the same whether other forces act on the body or not, the same as if it alone acted on the body at rest. This law enabled us in Chapter II. to deduce the Parallelogram of Forces from the Parallelogram of Velocities, and from the Parallelogram of Forces we have worked through the whole subject of statics. For this reason it was stated in Article 34 that of the three laws the second was, for present purposes, by far the most important.

Not only does the second law enable us to measure force but it also enables us to measure mass. For if $F = ma$, $m = \frac{F}{a}$; consequently, if the same force act on bodies of different mass, these masses must be inversely proportional to the accelerations which are communicated to them by the force.

In the present chapter the equation $F = ma$, which states the second law in an algebraical form, will be applied to the working out of various questions depending on the relations which exist between the force which acts, the mass of the particle it acts upon, and the change of velocity which is thereby produced.

192. Thus we shall find that the second law is sufficient to determine the motion of a single particle when it is acted upon by any forces, as it has also been sufficient to deduce the conditions of equilibrium of a number of forces in statics; but when we come to consider the motion of a system consisting of two or more particles we shall find that this knowledge is not sufficient for us. The additional knowledge required is then supplied to us by the third law, which tells us that the effect of A on B is always equal and opposite to the effect of B on A.

It has been pointed out in the first chapter (Art. 4) that force is always of the nature of a *stress*—that is to say, there can be no force without an equal and opposite reaction.

Now since force is measured by the change of momentum which it can produce, we see that if the momentum of any part of a system be changed without the action of a force external to the system there must be an equal and opposite change of momentum communicated to the rest of the system. This condition must hold all through the system. It therefore follows that the sum of the momenta of the various parts of a system, parallel to any given direction, must remain constant so long as there is no external action, whatever be the mutual action between the various parts. This is called the principle of **Conservation of Momentum**. Hence it follows that the **centre of mass** of a system of mutually influencing particles obeys the first law of motion—that is to say, it **moves uniformly** unless acted on by some force external to the system.

193. This law is of the very greatest importance, although its full value cannot be appreciated by the student until he has advanced beyond the limits of this book. To the beginner the second law ($F = ma$) seems of paramount importance, for it enables him to deduce the conditions of equilibrium of a system of forces, and it enables him to determine the motion of a particle acted on by any forces. But in nature we more often have to deal with motion than rest, and we never meet with particles; consequently the results achieved by means of the second law are of small practical importance until they are developed by means of the third law.

As examples the student may consider the following :—

When a gun is fired off, the kick on the shoulder is measured by the momentum generated in the shot.

When two moving bodies meet, the momentum gained by one is equal to the momentum lost by the other—*i.e.* there is no change of momentum.

If two bodies, connected by a string, be projected into space, gravity being the only force, the motion of their centre of mass is a parabola, though the paths of the bodies are very various.

The attraction of the sun for the earth is equal and opposite to the attraction of the earth for the sun, and similarly

for the mutual action between any two of the planets of the solar system ; hence, although the motion of the centre of mass of the solar system is uniform, the motion of each planet is very complicated.

194 To deduce the equation $F = m a$ from the second law of motion.

Although *time* is not mentioned in the law, it is obvious that in estimating the effect of the force we must take into consideration the time during which it acts.

Let u, v be the velocities of the mass m at the beginning and end of time t , F the force. Then, change of momentum $\propto$ force,

$$\begin{aligned} m v - m u &\propto F \cdot t, \\ \frac{m(v - u)}{t} &\propto F, \\ m a &\propto F. \end{aligned}$$

We now choose $F = 1$, when $m = 1$ and $a = 1$; that is, we take our unit force to be the force which, acting on the unit of mass, produces in it the unit acceleration ;

$$\begin{aligned} \text{then } \frac{m \cdot a}{1 \cdot 1} &= \frac{F}{1} ; \\ \therefore m a &= F. \end{aligned}$$

195. Taking the unit force to be the force which can produce the unit of momentum in the unit of time, deduce the equation $F = m a$.

Let M = number of units of momentum produced in the mass m in time t ; then

$$M = m v - m u.$$

Unit force produces 1 unit momentum in 1 unit of time ;

$\therefore F$ produces F units of momentum in 1 unit of time ;

$\therefore F$ „ $F \cdot t$ „ „ t „

But F „ M „ „ t units of time ;

$$\begin{aligned} \therefore F \cdot t &= M \\ &= m v - m u ; \\ F &= \frac{m(v - u)}{t} = m a. \end{aligned}$$

196. The equation $W = m g$ is a particular form of the equation $F = m a$.

W is the measure of the force of gravity on a body of mass m , g is the acceleration it produces in it.

Moving force = mass moved $\times$ acceleration produced,

$$W = m \times g.$$

Hence, taking the ft. lb. sec. units, weight of 1 lb. = g poundals; and, generally, the measure of a force in poundals = g times the measure of the force in lb. wt. units ($g = 32.2$).

So also, taking the C. G. S. units, the measure of a force in dynes = g times the measure of the force in gramme wt. units ($g = 980$).

197. *Two masses m and m' are connected by a light string passing over a smooth pulley. Find the acceleration of the system and the tension of the string.*

Let T be the tension of the string, a the acceleration of the system.

Suppose m greater than m' . Then m moves downwards with the same acceleration as m' moves upwards.

Moving force acting downwards on $m = m g - T$; moving force acting upwards on $m' = T - m'g$; $\therefore$ by the second law of motion

$$m g - T = m a \quad \dots \quad (1)$$

$$T - m'g = m a \quad \dots \quad (2)$$

Adding, we get $m g - m'g = (m + m') a$;

$$\therefore a = \frac{(m - m')g}{m + m'} \quad \dots \quad (3)$$

Substituting for a in equations (1) or (2) we get

$$T = \frac{2 m m' g}{m + m'} \quad \dots \quad (4)$$

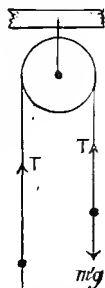


FIG. 135.

Note.—Equation (3) might have been obtained at once by considering the two masses as one mass $(m + m')$ acted on by a force equal to the difference between the two weights $(m g - m' g)$.

T the tension is greater than $m'g$ and less than mg . If the masses be equal there is no acceleration, and $T = \frac{2m^2g}{2m} = mg =$ the weight of either ; that is, the tension is the same as if the system had been at rest.

This example embodies the principle of Atwood's machine, which we will consider at the end of the chapter.

It will be noticed that we have neglected

- (1) the mass of the string ;
- (2) the inertia of the pulley ;
- (3) the friction of the pulley.

Of these (1) is inconsiderable if fine silk thread be used, but in practical work (2) and (3) must be taken into account.

Example.—Masses of 7 lbs. and 9 lbs. are connected by a string passing over a smooth peg ; find the acceleration of the system and the tension of the string. With the same notation

$$9g - T = 9a,$$

$$T - 7g = 7a;$$

$$\therefore 2g = 16a,$$

or $a = \frac{1}{8}g = 4$ feet per second per second.

$$T = 9g - 9a = 9(g - \frac{1}{8}g) = \frac{63}{8} \cdot g$$

$$= 252 \text{ poundals.}$$

198. *A mass m hanging freely is connected by a light string with a mass m' on a smooth table ; to find the acceleration a .*

In considering the motion of m' , we notice that T is the only force which acts on it in the direction of motion ;

$$\therefore T = m'a \quad \dots (1)$$

For the motion of m vertically downwards we have

$$mg - T = ma \quad \dots (2)$$

Adding (1) and (2) $mg = m'a + ma ;$

$$\therefore a = \frac{m}{m' + m} \cdot g.$$

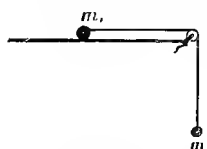


FIG. 136.

Hence it is obvious that, however small m may be, and however large m' may be, motion must take place provided that the table be perfectly smooth.

199. *If two masses m' and m be connected as in the preceding article, and it is observed that m' moves over s feet in t seconds, starting from rest, to determine the coefficient of friction between m' and the table.*

If F = force of friction on m' ,
 R = pressure of the table on m' ,
 a = acceleration of the system,

we have the following equations :—

$$\begin{aligned} T - F &= m' a \text{ for the motion along the table,} \\ R - m' g &= 0 \text{ for the motion } \perp \text{ to the table,} \\ F &= \mu R ; \end{aligned}$$

$$\therefore T - \mu m' g = m' a.$$

For the motion of m we have

$$m g - T = m a ;$$

$$\therefore m g - \mu m' g = (m' + m) a ;$$

$$\therefore \mu = \frac{m}{m'} - \left(1 + \frac{m}{m'} \right) \frac{a}{g},$$

$$\text{and } a = \frac{2s}{t^2} ;$$

hence μ can be found.

It is obvious that m must be greater than $\mu m'$, or there will be no motion.

200. *To find the motion of a particle m free to move on a rough inclined plane of inclination θ .*

Let a = the acceleration,

R = the pressure of the plane on the particle,

F = the force of friction.

Resultant force $\perp$ to the plane = $R - m g \cos \theta$.

Resultant force down the plane = $m g \sin \theta - F$;

$$\therefore R - m g \cos \theta = 0,$$

and

$$mg \sin \theta - F = ma,$$

and

$$F = \mu R;$$

$$\therefore mg \sin \theta - \mu mg \cos \theta = ma;$$

$$\therefore a = (\sin \theta - \mu \cos \theta) g.$$

We note that $\tan \theta$ must be greater than μ .

Similarly, if the particle were projected up the plane with any velocity u , the retardation down the plane would be

$$(\sin \theta + \mu \cos \theta) g,$$

and it would come to rest after passing over a distance

$$= \frac{u^2}{2g(\sin \theta + \mu \cos \theta)}.$$

201. A mass m' resting on a smooth inclined plane of inclination θ' is connected with a mass m by a light string passing over a pulley at the top of the plane so that m hangs freely; to find the acceleration a .

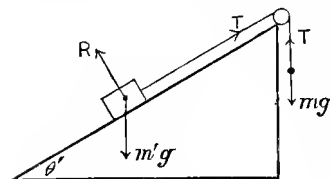


FIG. 137.

(1) If the plane be smooth, considering the motion of m' , since there is no acceleration perpendicular to the plane, the forces perpendicular to the

plane must be in equilibrium; $\therefore R = m'g \cos \theta'$.

The resultant force up the plane $= T - m'g \sin \theta'$;

$$\therefore T - m'g \sin \theta' = m'a \quad \dots \quad (1)$$

$$\text{Moving force on } m = mg - T = ma \quad \dots \quad (2)$$

$$\therefore mg - m'g \sin \theta' = (m + m')a$$

$$a = \frac{m - m' \sin \theta'}{m + m'} \cdot g.$$

If m be less than $m' \sin \theta'$, the motion will be down the plane.

(2) If the plane be rough.

Let μ = the coefficient of friction,

F = the force of friction opposite to the direction of motion.

Then $F = \mu.R = \mu m' g \cos \theta'$.

Hence equation (1) becomes

$$T - m' g \sin \theta' - \mu m' g \cos \theta' = m' a ;$$

$$\therefore a = \frac{m - m'(\sin \theta' + \mu \cos \theta')}{m + m'} \cdot g.$$

If the acceleration were down the plane F would act in the same direction as T , and the equations of motion would be—

$$m' g \sin \theta' - F - T = m' a,$$

$$T - m g = m a ;$$

$$\therefore m' g \sin \theta' - F - m g = (m + m') a ;$$

$$\therefore a = \frac{m'(\sin \theta' - \mu \cos \theta') - m}{m + m'} \cdot g.$$

Hence, for motion on the plane either up or down, m must not lie between

$$m'(\sin \theta' - \mu \cos \theta') \text{ and } m'(\sin \theta' + \mu \cos \theta').$$

This result coincides with the result obtained in Article 134.

If both m and m' be on rough planes whose inclinations are θ and θ' , the coefficient of friction being μ , the student will easily obtain the following expression for a when m is moving down the plane

$$a = \frac{m(\sin \theta - \mu \cos \theta) - m'(\sin \theta' + \mu \cos \theta')}{m + m'} \cdot g.$$

202. *To find the pressure of a mass m which is at rest on a horizontal plane which itself has a vertical acceleration a .*

Let R be the pressure between the body and the plane, then $R - m g$ is the resultant moving force upwards.

Hence, if a = the vertical acceleration,

$$R - m g = m a ;$$

$$\therefore R = m(g + a).$$

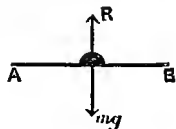


FIG. 138.

Similarly, if the motion be downwards,

$$m g - R = m a ;$$

$$\therefore R = m(g - a).$$

If W be weight of the body, $W = mg$;

$$\begin{aligned}\therefore R &= \frac{W}{g} (g \pm a) \\ &= W \left(1 \pm \frac{a}{g} \right)\end{aligned}$$

according as the motion is up or down.

Atwood's Machine.

203. This is a machine constructed for the purpose of illustrating the first two laws of motion ; it can also be used for making a rough calculation of the value of g .

Referring to equation (3) of Article 197 we have,

$$mg - m'g = (m + m')a;$$

$m + m'$ may be kept constant, while $m - m'$ may be made to take any value less than $m + m'$. We may thus determine the accelerations produced by different forces on the same mass,

Force $\propto$ acceleration.

Again, $m - m'$ may be kept constant, while $m + m'$ may be made to have any value we please. We may thus compare the effects of the same force on different masses,

$$\text{Acceleration} \propto \frac{1}{\text{mass}}.$$

Again, the equation may be written

$$g = \frac{m + m'}{m - m'} \cdot a.$$

We can choose m and m' so that a may be made so small as to enable us to determine it by observation ; we can thus determine the value of g .

204. In using the machine we take two equal weights, each of mass m , and fasten them to the ends of a fine thread passing over a pulley in which the friction is reduced to a minimum by letting the extremities of its axis rest between friction wheels. (See Deschanel's 'Natural Philosophy'.)

To one of the equal masses m a rider of mass m' is

attached ; this at once makes the system begin to move with an acceleration

$$a = \frac{m' g}{2 m + m'}.$$

The distances traversed by the ascending or descending mass in any time can be observed by means of a graduated scale placed in close proximity to their path.

The rider can be detached at any time by making the descending mass pass through a ring R, through which the rider cannot pass. After the rider is detached the acceleration vanishes and the velocity becomes uniform.

The distance traversed with this uniform velocity can now be observed for any given time, and hence the velocity can be calculated.

Thus, suppose the descending mass to start from rest at A, to move with uniform acceleration a to R, when the rider is detached, and then to move with uniform velocity v to S. We can make the following observations :—

$$\text{Distance A R} = a ;$$

$$,, \quad \text{R S} = b ;$$

$$\text{Time from A to R} = t ;$$

$$,, \quad \text{R to S} = t'.$$

By varying t we can prove that $a \propto t^2$, and that $v \propto t$.

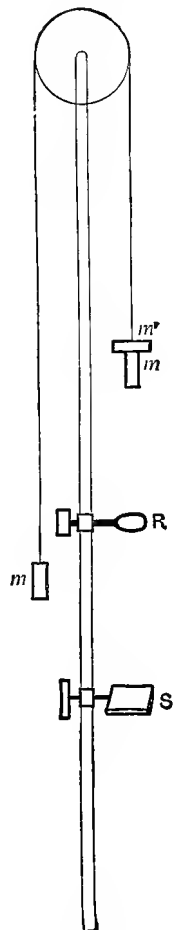


FIG. 139.

$$\text{We have also from our formulæ } a = \frac{1}{2} a \cdot t^2 \quad . \quad . \quad . \quad (1)$$

$$v = a t \quad . \quad . \quad . \quad (2)$$

$$v^2 = 2 a \cdot a \quad . \quad . \quad . \quad (3)$$

$$b = v \cdot t' \quad . \quad . \quad . \quad (4)$$

Theoretically a can be calculated from (1), $a = \frac{2a}{t^2}$; or from the equation, $b = a \cdot t \cdot t'$. Practically it is easier to measure t' than t ; we therefore eliminate v between equations (3) and (4); thus

$$\begin{aligned}\frac{b^2}{t'^2} &= 2a \cdot a; \\ \therefore a &= \frac{b^2}{2a t'^2}; \\ \therefore g &= \frac{2m + m'}{m'} \cdot a \\ &= \frac{2m + m'}{m'} \cdot \frac{b^2}{2a t'^2},\end{aligned}$$

an equation from which we may determine g .

For a description of the apparatus used for measuring the time, see Deschanel's 'Natural Philosophy.'

For the allowances to be made for the friction and inertia of the pulley, see also Deschanel, or Glazebrook and Shaw's 'Practical Physics,' p. 137.

It is shown in the latter place that the above equation must be written

$$g = \frac{2m + W + m' - F}{m' - F} \cdot \frac{b^2}{2a t'^2},$$

where W and F can be determined by experiment.

205.* With the most delicate instruments that can be made the value of g thus obtained is only a rough approximation to its actual value, owing to many unavoidable sources of error.

The most accurate method of calculating the value of g at any particular spot on the earth's surface is derived from observations of the time of oscillation of a simple pendulum, which consists of a small heavy particle suspended from a fixed point by a fine thread.

Thus, if l be the length of the thread and t the time of a complete oscillation,

$$\begin{aligned}t &= 2\pi \sqrt{\frac{l}{g}} \quad . \quad . \quad . \quad . \quad . \quad (1) \\ \therefore g &= \frac{4\pi^2 l}{t^2};\end{aligned}$$

or if n be the number of complete oscillations in 1 second,

$$t = \frac{1}{n}; \quad \therefore g = 4\pi^2 n^2 l.$$

Both n and l are capable of the most exact measurement.

It is necessary for the student to defer the proof of equation (1) until he has advanced a good deal further in the subject.

Example 1.—A particle whose mass is 15 lbs. is at the centre of a smooth circular table whose radius is 4 feet, and is connected by a fine thread with a mass of 1 lb. hanging just over the edge.

If the height of the table be 3 feet, how long will the heavy particle take to get off the table?

Let T be the tension of the string, a the acceleration, then

$$T = 15a,$$

$$1 \cdot g - T = a;$$

$$\therefore 16a = g;$$

$$\therefore a = 2 \text{ feet per second per second.}$$

Let t be the time taken, v the velocity acquired in passing over 3 feet; then

$$3 = \frac{1}{2} a t^2, \text{ and } v^2 = 2 a \times 3;$$

$$\therefore t = \sqrt{3}, \text{ and } v = 2\sqrt{3}.$$

The string now becomes slack, and the particle moves over the remaining distance of 1 foot with uniform velocity v ;

$$\begin{aligned} \therefore \text{whole time taken} &= \sqrt{3} + \frac{1}{2\sqrt{3}} = \frac{7\sqrt{3}}{6} \text{ seconds} \\ &= 2 \text{ seconds approximately.} \end{aligned}$$

Example 2.—A sand bag of 50 lbs. is on the floor of a balloon, which is ascending with an acceleration of 8 feet per second per second. Determine its pressure on the floor.

Let R be the mutual pressure between the bag and the floor.

Then the moving force = mass moved $\times$ acceleration.

$$R - 50g = 50 \times 8,$$

$$R = 50(g + 8)$$

$$= 50 \times 40$$

$$= 2,000 \text{ poundals,}$$

$$\text{or } 62\frac{1}{2} \text{ lbs. wt.}$$

Example 3.—Two masses of 5 lbs. each are connected by a string and placed on a rough plane inclined to the horizon at an angle $\sin^{-1}\frac{3}{5}$, so that one just hangs over the top when the other is

at the bottom. If the coefficient of friction be $\frac{1}{4}$, and the height of the plane 12 feet, find the time taken by the freely hanging mass to reach the base of the plane, and determine how near the top the other mass will get.

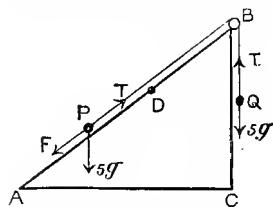


FIG. 140.

Let A B C be the plane,

$$\text{angle } B A C = \theta = \sin^{-1} \frac{3}{5};$$

$$\therefore \sin \theta = \frac{3}{5}, \text{ and } \cos \theta = \frac{4}{5}.$$

Since B C = 12 feet,

$$A B = \frac{5}{3} \times 12 \text{ feet} = 20 \text{ feet.}$$

Calling the particles P and Q, when Q is at C, P will be at D, a point 8 feet from B.

If F be the force of friction, and R the pressure between P and the plane,

$$R = 5g \cos \theta = 5g \times \frac{4}{5} = 4g \text{ poundals};$$

$$\therefore F = \mu R = \frac{1}{4} \times 4g = g \text{ poundals.}$$

Let T be the tension of the string, a the acceleration, then the moving force on P up the plane

$$= T - 5g \sin \theta - F$$

$$= T - 3g - g$$

$$= T - 4g.$$

$$\therefore T - 4g = 5a.$$

For the motion of Q

$$5g - T = 5a;$$

$$\therefore g = 10a.$$

Let v be the velocity acquired, and t the time taken in passing over 12 feet from rest with an acceleration $\frac{g}{10}$.

$$\text{Then } 12 = \frac{1}{2} \frac{g}{10} \cdot t^2;$$

$$\therefore t^2 = \frac{12 \times 20}{g} = 7.5,$$

$$t = 2.7 \text{ seconds.}$$

After this time Q reaches the ground, and P passes the point D.

$$v^2 = 2 \cdot \frac{g}{10} \cdot 12$$

$$= \frac{24g}{10}.$$

As P passes D the string becomes slack, consequently P continues the ascent with an initial velocity v , which is subject to a retardation due to the friction of the plane, which is $= \frac{F}{m} = \frac{1}{5}g$, and also to the retardation of gravity resolved down the plane.

Let s = distance travelled by P above D before it comes to rest,

then

$$0 = v^2 - 2 \cdot \left(\frac{g}{5} + \frac{3g}{5} \right) s;$$

$$\therefore s = \frac{5v^2}{8g} = \frac{5 \times 24g}{8g \times 10} = 1\frac{1}{2} \text{ ft.};$$

$\therefore$ P gets within $6\frac{1}{2}$ feet from the top of the plane.

Example 4.—P, Q and R are three masses, of which P and Q are connected by a light string passing over a smooth pulley, and R is attached to Q by another light string of length l ; P is $>Q$ and $<Q+R$. Motion is allowed to commence when Q is on the point of leaving the plane on which R rests; to find the time that will elapse between the time that R leaves the plane and strikes it again.

Acceleration of P and Q $= \frac{P-Q}{P+Q} \cdot g$.

Let v = the velocity acquired in passing through the distance l ,

$$v^2 = 2 \frac{P-Q}{P+Q} \cdot gl.$$

Let v' = velocity with which R leaves the plane; then, since the momentum immediately after the addition of R = momentum before,

$$v'(P+Q+R) = (P+Q)v,$$

$$v' = \frac{P+Q}{P+Q+R} v.$$

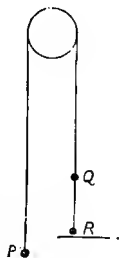


FIG. 147.

Acceleration of the system is now opposite to the direction of motion and $= \frac{Q+R-P}{Q+R+P} \cdot g = a'$.

Time in which v' is destroyed by a retardation $a' = \frac{v'}{a}$;

$$\therefore \text{time of R's flight} = \frac{2v'}{a'}.$$

$$\begin{aligned}
 \therefore \text{time} &= 2 \frac{P+Q}{P+Q+R} \cdot v \times \frac{Q+R+P}{(Q+R-P)g} \\
 &= \frac{2(P+Q)}{Q+R-P} \cdot \frac{v}{g} \\
 &= \frac{2(P+Q)}{Q+R-P} \cdot \sqrt{2 \frac{P-Q}{P+Q} \cdot \frac{l}{g}} \\
 &= \frac{2\sqrt{2}\sqrt{P^2-Q^2}}{Q+R-P} \sqrt{\frac{l}{g}}
 \end{aligned}$$

Examples.

1. What is the force required to communicate to a mass of 36 lbs. a velocity of 10 yards a minute in a quarter of an hour?

2. Of two forces, one acts on a mass of 5 lbs. and produces in it a velocity of 5 feet per second in $\frac{1}{11}$ second, and the other acts on a mass of 625 lbs., and produces in it a velocity of 18 miles per hour in one minute. Compare the two forces.

3. A particle of 10 lbs. is placed on a plane which is made to descend with a uniform acceleration of 10 feet per second per second. Find the pressure on the plane.

4. A particle of mass m is moving with a constant velocity v along one of the sides of a square, and when it arrives at the end of that side it receives a blow which makes it move along the adjacent side of the square with the same velocity v . Find the momentum imparted by the blow.

5. A mass of 1 lb is laid on a smooth table and connected by a string with a mass of 1 oz., which hangs freely. Determine the acceleration.

6. A mass of 112 lbs. is placed on a smooth horizontal table with its centre of gravity one foot from the edge of the table; to it is attached a string which is also attached to a mass of 14 oz., hanging over the edge of the table. How long will it be before the latter pulls the larger mass off the table?

7. A mass of 10 lbs. rests on a rough horizontal table, and is connected by a string passing over a pulley at the edge of the table to a mass of 3 lbs. hanging freely. If the coefficient of friction $= \frac{1}{5}$, find the acceleration and the tension of the string.

8. Masses of $23\frac{1}{2}$ lbs. and $24\frac{1}{2}$ lbs. are attached to the ends of a string which passes over a smooth pulley. Find (1) the accelera-

tion of the system ; (2) the space described in four seconds from rest ; (3) the space described in the fourth second.

9. In Atwood's machine the equal masses are each 1 lb. and the small mass 1 oz. Find the space described in $2\frac{3}{4}$ seconds after the motion commences ; and if the small mass be then taken off, find the space described by the others in the next second. Compare the tensions of the string before and after the 1 oz. is taken off.

10. Masses of 1 lb. and $1\frac{1}{4}$ lb. are connected by a string passing over a smooth pulley. Find the tension of the string and the distance moved from rest in three seconds.

11. If the heavier of two masses suspended at the ends of a string which passes over a pulley descends $12\frac{1}{2}$ feet in $2\frac{1}{2}$ seconds, and if when 1 oz. is added to each it only descends 10 feet in the same time, find the masses.

12. The equal masses in an Atwood's machine are each $12\frac{1}{2}$ oz. ; the mass of the rider is $1\frac{2}{3}$ oz. Find the acceleration.

13. Two equal bodies A and B, each of mass M, are connected by a string passing over a smooth pulley. To A a body C of mass m is attached. A and C are started upwards with velocity v , B starting downwards with the same velocity. How much string will pass over the pulley before the system comes to rest ?

14. Two equal masses, the one of iron, the other of brass, are connected by a string over a frictionless pulley. A magnet is placed below the iron so as to attract it. Show that the attractive force is measured by the mass of either multiplied by the space descended by the iron in two seconds, the force of the magnet being supposed constant during the motion.

15. Masses M and m hang at the ends of a string which passes over a smooth pulley. If M strike the ground one second after motion commences, how much longer will it be before m first comes to rest, and what will its distance then be from the ground if the two masses were originally at the same level ?

16. A mass M is connected with a mass $2M$ by means of a string which passes over a smooth pulley, the strings being vertical. The first mass falls freely for one second and then begins to lift the second mass from rest. Find how high the latter will rise, and the time that elapses before it returns to its original position.

17. Two weights are connected by a string passing over a

smooth pulley. In the fourth second of the motion seven feet of string pass over the pulley. Find the ratio of the weights.

18. Find the force of friction acting upon a train whose mass is 1,000 tons which, starting up an incline of 1 in 50 with a velocity of 30 miles per hour, is brought to rest in 300 yards.

19. A horse continually exerts a pull equal to the weight of 150 lbs. upon a truck whose mass is 5 tons placed on smooth level rails. The maximum speed of the horse is 10 miles an hour; how long is it before this speed is attained?

20. A force equal to the weight of 10 lbs. is required to make a mass of 10 lbs. move with uniform velocity up a plane board whose inclination to the horizon is 45° . Find the coefficient of friction.

21. In the preceding question, if the force be equal to the weight of 11 lbs., determine the acceleration.

22. A train is travelling at the rate of 45 miles an hour on level rails when steam is shut off. Assuming that $g = 32$, and that the resistance of friction and the air to the train's motion is 11 lbs. weight per ton, find how far it will run, and for what time, before it comes to rest.

23. A mass m resting on a smooth plane of inclination 30° is connected by a light string passing over a pulley at the top to an equal mass hanging freely. Find the acceleration and the tension of the string.

24. What must be the masses attached to the ends of the string of an Atwood's machine, and the mass of the rider, in order that the action of a force of 10,000 dynes upon a mass of 1,000 grammes may be investigated. (Acceleration due to gravity is 981 cm. per sec. per sec.)

25. Two masses of 5 oz. and 4 oz. are connected by a string over a frictionless pulley. A rider of 1 oz. is supported by a ring one foot above the initial position of the smaller weight so that it is lifted off as the weight passes the ring. If the masses start from rest, find the uniform velocity of the system after the rider is lifted off.

26. Masses of 7 lbs. and 9 lbs. are connected as in Atwood's machine. The system starts from rest; after moving through 2 feet 1 lb. is detached from the 9 lbs. Find the whole distance travelled by the masses in three seconds from their position of rest.

27. In the above question, if 3 lbs. were detached from the 9 lbs., find the time at which the masses would be at rest again.

28. Find the space through which two masses of 7 and 8 lbs., connected as in Atwood's machine, would move in four seconds, starting from rest, if, after the system had moved through 5 inches, 1 lb. was taken from the larger weight.

29. If the weights in Atwood's machine were equalised by the smaller mass picking up a rider, instead of the larger one losing a rider, how would you modify your result for the velocity after the rider had been picked up?

In the preceding question, find the velocity of the system if, after motion through 5 inches, the 7 lb. mass picks up a rider of 1 lb.

30. How would a man, ascending in a balloon, determine his upward acceleration by means of a spring balance?

31.* A weight W hangs over a pulley. A monkey takes hold of the other end, and when W is at rest commences to climb, and climbs up a height h in t seconds without disturbing W . Find his weight. If at the end of t seconds he ceases to climb, how much further will he ascend in the next t seconds?

32.* A light string has masses P and Q attached to its ends, and is put over two fixed pulleys, the portion between them supporting a movable pulley of mass R . Write down the equations of motion for P , Q , and R , and find the fourth equation necessary to determine the motions.

CHAPTER XV

HODOGRAPH—NORMAL ACCELERATION—MOTION IN A CIRCLE

206. IN Chapters XI., XII., and XIV. we have considered the case of a change of velocity in the direction of motion and the force which produces such change, the motion being in a straight line.

In Chapter XIII. we have considered the case of a constant force always acting in the same direction but not in the direction of motion, and we have seen that the path of the body so acted upon is a parabola. In this case there is a change of velocity both in magnitude and direction.

Generally, when a body is moving under the action of any force which does not act in the direction of motion, the force may be resolved into two components at right angles to each other—one along the tangent to the path, and the other along the normal ; the former is in the direction of motion, and produces a change in the *magnitude* of the velocity only ; the latter is perpendicular to the direction of motion, and produces a change only in the *direction* of the velocity.

If the latter vanishes there is no change in the direction, and the path is a straight line ; if the former vanishes, the path is curved, but the speed does not alter.

In the present chapter we shall first consider the case of a particle moving with uniform speed in a circle ; the force producing the change of motion will always be normal to the path—that is, towards the centre.

In order to determine the normal acceleration in such a case, we will first define and consider the properties of a curve called the Hodograph.

Definition.—If a point P be moving in any manner along a curve and if from a fixed point O a straight line OQ be drawn representing the velocity of P in magnitude and direction, the locus of Q is called the **hodograph** of the path of P .

207. We will now prove that the velocity of Q in the hodograph represents in magnitude and direction the acceleration of P at the corresponding point of the path

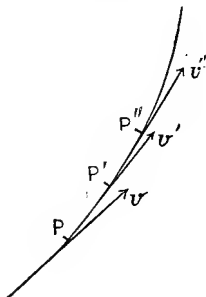


FIG. 142.

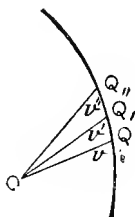


FIG. 143.

Let P, P', P'' be consecutive points in the path of a particle close to each other, at which the velocities are v, v', v'' respectively; from O draw OQ, OQ', OQ'' parallel and numerically equal to v, v', v'' respectively; then Q, Q', Q'' are the points on the hodograph corresponding to P, P', P'' .

Then, if τ be the indefinitely small time required to describe the arc PP' , when PP' is indefinitely small,

$$v = \frac{PP'}{\tau}. \quad (\text{See Definition, Art. 15.})$$

Now OQ = velocity of P at beginning of time τ ,
and OQ' = velocity of P at end of time τ ;
therefore QQ' = change of velocity in time τ . (Art. 153.)

Hence velocity of Q in the hodograph

$$\begin{aligned} &= \frac{QQ'}{\tau} = \frac{\text{change of } P\text{'s velocity in time } \tau}{\tau} \\ &= \text{acceleration of } P. \end{aligned} \quad (\text{Art. 17.})$$

Again, just as the limiting position of the chord PP' , when PP' is indefinitely small, becomes the tangent at the point P , and is the direction of the velocity at P , so the limiting position of QQ' is the tangent to the hodograph at the point Q , and consequently represents the direction of the acceleration of P .

Hence the velocity of the point Q in the hodograph represents the acceleration of P in magnitude and direction.

208. To determine the normal acceleration a of a particle describing a circle of radius r with uniform speed v .

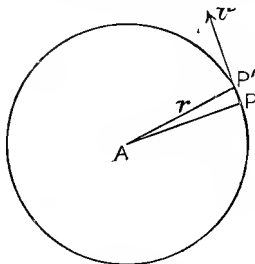


FIG. 144.

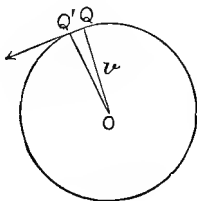


FIG. 145.

Let P, P' be contiguous positions of the particle in its circular path; since P 's rate of motion is uniform, its hodograph is a circle whose radius is v . Let O be the centre of this circle, Q, Q' points on it corresponding to the points P and P' on the path.

Then PP' , the tangent at P , is the direction of P 's velocity; and QQ' , the tangent at Q , is the direction of Q 's velocity—that is, of P 's acceleration; but QQ' is perpendicular to OQ , and OQ is parallel to PP' ; therefore, QQ' is perpendicular to PP' ; hence, QQ' is parallel to PA —that is, *the acceleration of P is inwards towards the centre of the circle.*

Again, since P describes a circle with uniform speed v , Q will describe the hodograph with uniform speed a in the same time.

Hence, the speeds are proportional to the radii;

$$\therefore a : v :: v : r;$$

$$\therefore a = \frac{v^2}{r}.$$

Hence, when a particle describes a circle of radius r with uniform speed v , there is always a uniform acceleration towards the centre of the circle equal to $\frac{v^2}{r}$.

If the mass of the particle be m , the force necessary to produce this acceleration is $\frac{mv^2}{r}$.

209. The following are examples of this force : the tension of the string attached to a particle which is being whirled round in a circle ; the pressure from the outside of a circular groove on a marble which is travelling round it ; the pressure from the inside surface of the outside rail of a railway track on the flanges of the wheels of a train travelling round a curve. It is a force which is familiar to everybody ; when a boy is running quickly round a sharp curve, he knows that he has to incline his body towards the centre of the curve, and put his feet on the ground in such a way that a great amount of lateral friction may be brought into action. We will now calculate the magnitude of the force exerted in such a case.

Example.—A boy of 9 stone is running round a curved running-path whose radius is 121 feet at a rate of 15 miles an hour ; find the normal force necessary to keep him in his circular path.

$$\begin{aligned}\text{Normal force} &= \frac{mv^2}{r} = \frac{9 \times 14 \times (22)^2}{121} \text{ poundals} \\ &= 504 \text{ poundals} \\ &= \text{weight of } 16\frac{3}{4} \text{ lbs. nearly.}\end{aligned}$$

It is this force which causes the moon to describe its curvilinear orbit round the earth ; in this case the force is due to the attractive or gravitative action between the two bodies.

Formerly this force received the name of ‘centrifugal force,’ as though a body desired to flee from the centre ; later this was changed by some writers to ‘centripetal force,’ as though a body wished to reach the centre. Both terms are misleading, since a body has no wish to do the one or the other ; if left to itself it would continue to move with uniform velocity in a straight line, and, therefore, if its path is curvilinear it must be on account of the action of some force along

the normal. We may call it the normal force, or the force required to produce the normal acceleration ; its measure is always $\frac{mv^2}{r}$.

If the path of the body is not circular, but is curvilinear, the normal acceleration is $\frac{v^2}{\rho}$, where ρ is the radius of curvature at any point—that is, the radius of the circle passing through *three* contiguous points on the curve. Hence, whatever be the path described by a body, we may look upon its motion at any point as consisting of two parts—an acceleration along the tangent equal to the limit of $\frac{v' - v}{\tau}$, when τ is indefinitely small, and an acceleration along the normal equal to $\frac{v^2}{\rho}$, where ρ is the radius of curvature at the point.

Note.—If the path be a straight line, $\rho = \text{infinity}$, and the expression for the normal acceleration vanishes.

210. A particle of mass m attached to a fixed point by a string of length l is projected so that it describes a circle in a horizontal plane with uniform speed v . To find T , the tension of the string, and its inclination θ to the vertical.

Let P be the fixed point, A the position of the particle at any time, O the centre of the horizontal circle which it is describing.

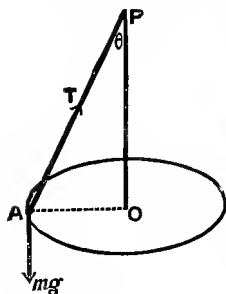


FIG. 146.

Let $OA = r$.

The only forces which act upon m are its weight mg vertically downwards, and T the tension of the string ; but the normal force necessary to make m describe the circle whose radius is r is $\frac{mv^2}{r}$; therefore $\frac{mv^2}{r}$ is the resultant of T and mg ;

$$\therefore T \sin \theta = \frac{mv^2}{r},$$

and

$$T \cos \theta = mg,$$

now

$$\sin \theta = \frac{r}{l};$$

$$\therefore T = \frac{mv^2 l}{r^2}.$$

From these equations we can find T and θ in terms of l and v .

The results may be given in a more convenient form if we take n as the number of revolutions per second. In this case

$$v = 2\pi r \cdot n;$$

$$\therefore T = \frac{ml(2\pi r \cdot n)^2}{r^2}$$

$$= 4ml\pi^2 \cdot n^2 \text{ poundals};$$

and

$$\cos \theta = \frac{mg}{T} = \frac{g}{4l\pi^2 n^2}.$$

If t = time of a complete revolution, $t = \frac{1}{n}$;

$$\therefore t^2 = \frac{4\pi^2 l \cos \theta}{g};$$

$$\therefore t = 2\pi \sqrt{\frac{PO}{g}}.$$

The above is known as the **Conical Pendulum**.

211. By an extension of Article 172 of Chapter XII. we can prove that *the velocity acquired by a particle in sliding down a smooth curve is exactly the same as that which would be acquired by a particle in falling through the same vertical height.*

Thus, let P, P' be contiguous points on the curve, and let PP' the tangent at P make an angle θ with the horizontal; let v, v' be the velocities at P and P' respectively: then for an indefinitely short space of time we may consider that the particle is moving through the space PP' with an acceleration $g \sin \theta$;

$$\begin{aligned}\therefore v'^2 &= v^2 + 2g \sin \theta \cdot PP' \\ &= v^2 + 2g \cdot PN,\end{aligned}$$

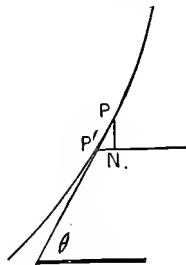


FIG. 147.

and so on all the way down the curve. Hence, if v be the final velocity, u the initial velocity, and h the vertical height,

$$v^2 = u^2 + 2gh.$$

Similarly, if a particle be projected up a smooth curve, $v^2 = u^2 - 2gh$.

These results are exactly the same if the motion of the particle is constrained by means of a string instead of a smooth surface; in all such cases $v^2 = u^2 \pm 2gh$, according as the motion is with or against gravity.

The results of this Article can also be obtained directly by the method of Chapter XVII.

212. *A particle m is attached to a string of length l , and is whirled round in a vertical circle; to find the tension of the string at any point, and the condition that m shall just describe a complete circle.*

We will suppose that it passes the lowest point A with a velocity u ; let P be its position at any subsequent time, v its velocity, h the vertical height above its lowest point, θ the angle which the string makes with the vertical. Then,

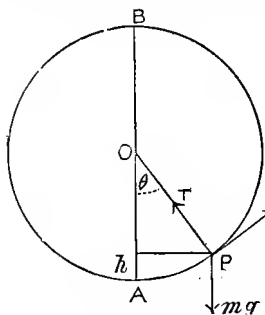


FIG. 148.

$$v^2 = u^2 - 2gh. \quad (1)$$

$$\cos \theta = \frac{l-h}{l}. \quad (2)$$

The normal force at P required to make the particle move in a circle $= \frac{mv^2}{l}$ along PO ; but the only forces which act on the particle are T along PO and mg vertically downwards;

$$\therefore \frac{mv^2}{l} = T - mg \cos \theta \quad (3)$$

The other component of mg (i.e. $mg \sin \theta$) is the cause of the tangential acceleration $-g \sin \theta$.

Substitute from (1) and (2) in (3);

$$\begin{aligned}
 \therefore T &= \frac{m(u^2 - 2gh)}{l} + mg \left(\frac{l-h}{l} \right) \\
 &= \frac{m}{l} \{u^2 - 2gh + gl - gh\} \\
 &= \frac{m(u^2 + gl - 3gh)}{l}.
 \end{aligned}$$

At B the highest point $h = 2l$;

$$\begin{aligned}
 \therefore T &= \frac{m(u^2 + gl - 6gl)}{l} \\
 &= \frac{m(u^2 - 5gl)}{l}.
 \end{aligned}$$

In order that the particle should just describe a complete circle, the string must never be slack until it just reaches the highest point. Hence, putting $T = 0$, we have $u^2 = 5gl$ —that is, the velocity with which it passes the lowest point must not be less than $\sqrt{5gl}$; in this case the velocity with which it passes the highest point $= \sqrt{gl}$.

If we substitute $u^2 = 5gl$ in the above expression for T , we get

$$T = \frac{m(6gl - 3gh)}{l}.$$

Putting $h = 0$ we have $T = 6mg$ at the lowest point.

Hence, if a particle is being whirled round in a circle, the tension of the string at the lowest point cannot be less than six times the weight of the particle.

Example 1.—A mass of 7 lbs. is tied to one end of a string $3\frac{1}{2}$ ft. long and is made to revolve round the other end on a smooth horizontal table once in a second. Find the tension of the string.

Here $v = 2\pi r = 2 \times \frac{22}{7} \times \frac{7}{2} = 22$;

$$T = \frac{mv^2}{r} = \frac{7(22)^2}{\frac{7}{2}} = 968 \text{ poundals.}$$

Example 2.—At what rate must a stone be projected horizontally in order that it may revolve round the earth, the earth

being supposed spherical and the resistance of the air being neglected?

In this case the normal force = mg .

Therefore, if v = velocity of projection and r the radius of the earth,

$$mg = \frac{mv^2}{r};$$

$$\therefore v^2 = rg,$$

$$v = \sqrt{rg}.$$

If t = time it takes to get round,

$$t = \frac{2\pi r}{v} = 2\pi \sqrt{\frac{r}{g}}, \text{ which is less than } 1\frac{1}{2} \text{ hour.}$$

Examples.

1. A mass of 1 lb. is tied to a string 1 yard long which cannot support a weight greater than that of 5 lbs. It is then whirled round in a horizontal circle on a smooth table. Show that it cannot perform more than sixty-nine complete revolutions in a minute.

2. A mass of 2 lbs. describes circles in a horizontal groove of 2 feet radius at the rate of 10 a minute. Find its pressure on the side of the groove.

3. Why is it harder work for a horse to pull a tramcar round a curve than on a straight track?

4. Show that by raising the outside rail of a railway track in going round a curve the tendency of the train to leave the rails is diminished.

Show that if θ be the inclination of the floor of the carriage to the horizontal when there is no lateral pressure, $\tan \theta = \frac{v^2}{gr}$, where r is the radius of the curve and v the velocity of the train.

Hence show that on a 5-foot track, round a curve of one-eighth of a mile radius, that for a mean velocity of 30 miles an hour the outside rail ought to be raised between 5 and 6 inches above the level of the inside rail.

5. A particle slides down from the highest point of a smooth sphere of radius r . At what height will contact with the sphere cease?

6. The direction of a projectile makes an angle θ with the hori-

zontal. Show that the radius of curvature at the point $= 2h \sec \theta$, where h is the distance of the point from the directrix. .

Verify this with the geometrical expression $\frac{2SP^2}{SY}$.

7. Prove that the hodograph of a point moving with uniform velocity is a point.

8. Prove that the hodograph of a projectile is a vertical straight line.

9. Show that if a conical pendulum makes a complete revolution in one second, the length of the axis of the cone traced out by the string is nearly 10 inches.

10. If a conical pendulum make four complete revolutions in $15\frac{2}{3}$ seconds when the string is 13 feet long and the particle describes a horizontal circle of 5 feet radius, find the value of g .

11. Show that, when a person whirls a weight at the end of a string, if he allows a little of the string to slip through his fingers the tension of the string is decreased.

12. If a particle whose mass is 1 oz. is whirled round in a vertical circle by a string 1 yard long so that the velocity at the lowest point is 24 feet per second, show that the greatest and least tensions are 14 poundals and 2 poundals respectively. Show that, whatever be the velocity at the lowest point, the difference between the greatest and least tension is 12 poundals.

13. A string 2 feet long, which cannot support more than the weight of 10 lbs., has a mass of 1 lb. attached to it, and is whirled round in a vertical circle. Show that the string will break if the velocity at any time exceeds 24 feet per second.

14. If l be the length of the string, and the particle be projected from the lowest point with a velocity $= \sqrt{4gl}$ the string will become slack when the particle is at a height $\frac{5l}{3}$ above the ground.

15.* In the preceding question, if the length of the string be $\sqrt{5}$ feet, show that the particle will reach the horizontal plane through the lowest point at a point whose distance from the lowest point is about four-fifths of an inch.

CHAPTER XVI*

IMPULSE—IMPACT

213. WE have seen that the effect of a force F acting on a mass m during a finite time t is measured by the change of momentum in the direction of the force ; thus

$$F \cdot t = mv - mu,$$

where u and v are the velocities of m in the direction of the force before and after its action ; or

$$F \cdot t = M,$$

where M is the change of momentum in the direction of the force.

Now if M be constant and finite, $F \cdot t$ must be constant and finite—that is to say that, as t decreases, F must increase, and if t is very small, F must be very large.

Now, in nature, a change of momentum is very often produced by what we call a *blow* or *impulse*, which is a force which acts for so short a time that we cannot measure it ; hence the force which produces the impulse is very great, but we cannot estimate its value since we cannot measure t .

Thus, suppose that t is the millionth part of a second, and that the change of momentum was unity ; then, since

$$F \cdot t = 1,$$

$$F = \frac{1}{t} = 1,000,000 \text{ poundals.}$$

We have no physical means of measuring such small fractions of time, consequently we cannot measure an impulse in terms of the unit of force—that is, in poundals.

We therefore leave the time out of consideration entirely, and take the following definition :—

An impulse is the effect of a force which produces a finite change of momentum in an indefinitely short space of time ;

or

$$I = mv - mu,$$

where I is the measure of the impulse. (Art. 31.)

This is the nature of the force when two moving bodies impinge on each other and then separate again ; when an explosion takes place, as, for instance, when a gun is fired off ; when a blow is given with a hammer, and so on.

214. *A particle of mass m is moving with velocity u , find the magnitude and direction of the impulse which will make it move with velocity v in a direction making an angle θ with its former direction.*

Let OA represent the velocity u , OB the velocity v .

Then AB represents the change in velocity ;

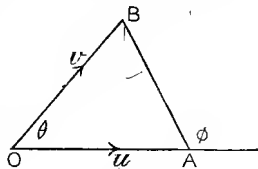


FIG. 149.

$\therefore m \cdot AB$ represents the impulse ;

$$\therefore \text{impulse} = m \sqrt{u^2 + v^2 - 2uv \cos \theta}.$$

So also the direction is ϕ where $\tan \phi = \frac{v \sin \theta}{v \cos \theta - u}$.

215. We will now consider the nature of the impact between two bodies.

The direction of the mutual impulse is called the *line of impact* ; it is the common normal, or perpendicular, to the portions of the surfaces of the two bodies which are in contact with each other at the instant of impact.

The effect of the impulse is to produce a change of momentum in each body in the line of impact only ; and by the third law of motion the momentum gained by the one body is equal to the momentum lost by the other—that is to say, the sum of the momenta of the two bodies in the line of impact is the same after impact as before.

If the direction of motion of each body be in the line of impact, then the impact is said to be *direct* ; but if the direc-

tion of either body be inclined to the line of impact, it is said to be *oblique*.

When two bodies come into contact, the particles in the neighbourhood of the point of contact are driven closer together by the impulse ; this is called the *force of compression*. It continues until the two bodies have a common velocity in the line of impact, after which the particles cannot be compressed further.

If now the particles of the substance of which the bodies are composed have no tendency to return to their former positions, there is no further impulse ; consequently the two bodies continue to move with a common velocity in their line of impact. Such bodies are termed *inelastic*.

In general, however, matter is not inelastic, but the particles of which it is composed, after having been compressed, tend to return to their former positions, and this tendency generates a *force of restitution*, which again separates the two bodies.

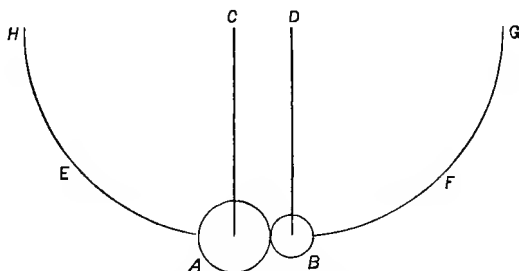


FIG. 150.

Newton made a series of experiments with spheres of different materials, by suspending them by strings in such a manner that the two spheres should impinge against each other at their lowest points. Their velocities at the moment of impact can be determined from the heights through which they have fallen, their velocities after impact from the heights to which they rise.

As the result of these experiments he was able to state the following law :—

When two bodies impinge on each other their relative velocity after impact bears a constant ratio to their relative velocity before impact, but is in the opposite direction.

This law, of course, only refers to the relative velocity in the line of impact; the velocities perpendicular to the line of impact remain unaltered.

This constant ratio is usually called the 'coefficient of elasticity,' or merely the 'elasticity'; sometimes it is called the 'coefficient of restitution'; it is usually denoted by the letter e .

Thus, if $u u'$ be the velocities of two bodies before impact, $v v'$ the velocities after, measured in the line of impact,

$$v - v' = -e(u - u').$$

This law, when combined with the third law of motion, will give us sufficient data to determine the velocities of two bodies after impact when their velocities before impact are given.

In the following articles we will consider the impact of two spheres, or of a sphere with a fixed surface.

Note.—The line of impact of two spheres is the line joining their centres.

216. *To determine the velocity of a ball (m), of elasticity e , moving with velocity u after impact with a smooth surface.*

(1) If the impact be direct.

Let v be the velocity after impact measured in the same direction as u .

The line of impact is the radius of the ball produced through the point of contact with the surface. Then

$$v = -e u;$$

that is, the ball rebounds along the line of impact.

The measure of the impulse

$$\begin{aligned} &= m u - m v \\ &= m u + m e u \\ &= m u (1 + e). \end{aligned}$$

(2) If the impact be oblique.

Let θ be the angle its direction makes with the line of impact before it strikes; and ϕ be the angle its direction makes with the line of impact after it strikes.

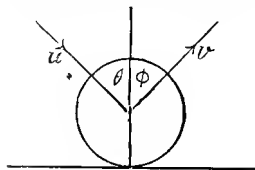


FIG. 151.

In this case let v be the velocity with which the ball leaves the surface; then, since the velocity perpendicular to the line of impact remains unaltered,

$$v \sin \phi = u \sin \theta.$$

By Newton's law $v \cos \phi = e u \cos \theta$.

Squaring and adding, $v^2 = u^2 (\sin^2 \theta + e^2 \cos^2 \theta)$; also

$$\tan \phi = \frac{\tan \theta}{e}.$$

The measure of the impulse $= m u \cos \theta (1 + e)$.

Note.—If the ball and the plane be inelastic $e = 0$.

The vertical velocity is destroyed, but the horizontal velocity remains the same; that is to say, the ball after impact will move *along* the plane with a velocity $u \sin \theta$.

Example.—A ball falls from a height of h feet on to a horizontal plane, the coefficient of elasticity being $\frac{1}{2}$; find the height to which it will rise after impact.

Let u = the velocity with which it strikes the plane;

v = " " leaves "

Then $u^2 = 2gh$, $v = eu = \frac{1}{2}u$.

Let x = the height to which it will rise; then

$$0 = v^2 - 2gx,$$

$$x = \frac{v^2}{2g} = \frac{u^2}{8g} = \frac{2gh}{8g} = \frac{h}{4}.$$

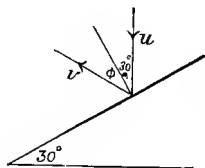


FIG. 152.

Example.—A ball of elasticity $\frac{1}{2}$ is dropped from a height of 100 feet on to a plane whose inclination to the horizon is 30° ; find the velocity and the direction in which it leaves the plane.

Let u = the velocity with which it reaches the plane, v = the velocity with which it leaves it, making an angle ϕ with the line of impact.

Then

$$u^2 = 2g \cdot 100 = 6,400;$$

$$\therefore u = 80;$$

$$v \sin \phi = u \sin 30^\circ = 80 \times \frac{1}{2} = 40,$$

$$v \cos \phi = e u \cos 30^\circ = \frac{1}{3} \times 80 \times \frac{\sqrt{3}}{2} = \frac{40}{\sqrt{3}};$$

$$\therefore v^2 \sin^2 \phi + v^2 \cos^2 \phi = (40)^2 \left(1 + \frac{1}{3}\right);$$

$$\therefore v^2 = \frac{(80)^2}{3};$$

$$\therefore v = \frac{80\sqrt{3}}{3};$$

$$\tan \phi = \frac{40}{\frac{40}{\sqrt{3}}} = \sqrt{3};$$

$$\therefore \phi = 60^\circ;$$

$\therefore$ its direction immediately after impact is horizontal.

Example.—What is the condition that the direction of an elastic ball after striking a horizontal plane should be perpendicular to its direction before impact?

With the notation of the preceding article, put $\theta + \phi = 90^\circ$;

then $\tan \phi = \frac{\tan \theta}{e};$

$$\therefore \cot \theta = \frac{\tan \theta}{e},$$

or $\tan^2 \theta = e;$

$$\therefore \tan \theta = \sqrt{e}.$$

This gives the direction of projection required. The necessary condition is independent of the velocity.

217. *Two inelastic spheres of mass m, m' , moving with velocities u, u' , impinge directly; to find the common velocity after impact.*

Let v be the common velocity.

By the third law of motion, sum of the momenta after impact = sum of the momenta before impact;

$$\therefore mv + m'v = mu + m'u';$$

$$\therefore v = \frac{mu + m'u'}{m + m'}.$$

If I be the measure of the mutual impulse,

$$\begin{aligned} I &= m u - m v = m' v - m' u' \\ &= \frac{m m' (u - u')}{m + m'}, \end{aligned}$$

by substituting for v the value just obtained.

218. *Two spheres, of elasticity e and mass m, m' , moving with velocities u, u' , impinge directly. To find their velocities after impact.*

Let v, v' be their velocities after impact.

Then, by Newton's experimental law,

$$v - v' = -e(u - u') \quad . \quad . \quad . \quad (1)$$

and by the third law of motion

$$m v + m' v' = m u + m' u' \quad . \quad . \quad . \quad (2)$$

Multiply (1) by m' , and add the equation thus formed to (2) :

$$(m + m') v = (m - e m') u + (m' + e m') u'.$$

Multiply (1) by m , and subtract the equation thus formed from (2) :

$$(m + m') v' = (m + e m) u + (m' - e m') u'. \quad]$$

These equations give v and v' .

If u' were in the opposite direction to u , we should write $-u'$ for u' all through the above equations.

219. *When two elastic spheres impinge directly on each other, to compare the impulse which causes the compression with the impulse due to the restitution of the particles.*

It has already been pointed out that the first part of the impact—the compression—goes on until the two bodies are moving with a common velocity, at which instant the force of restitution commences.

Let V be this common velocity, I the impulse of compression, I' the impulse of restitution.

Then $I = m u - m V = m' V - m' u'$;

whence $V = \frac{m u + m' u'}{m + m'}$, and $I = \frac{m m' (u - u')}{m + m'}$,

(as in Article 216).

Similarly, $I' = m V - m v = m' v' - m' V$;

whence $V = \frac{m v + m' v'}{m + m'}$, and $I' = \frac{m m' (v' - v)}{m + m'}$.

Substituting the values obtained for v and v' in the last article, we have

$$\begin{aligned} I' &= \frac{m m'}{(m + m')^2} \{e m u + e m' u - e m u' - e m' u'\} \\ &= \frac{m m'}{(m + m')^2} \times e (m + m') (u - u') \\ &= \frac{e m m' (u - u')}{m + m'} = e I; \end{aligned}$$

that is,

$$\frac{\text{impulse of restitution}}{\text{impulse of compression}} = e;$$

hence the name 'coefficient of restitution.'

The measure of the entire impulse

$$\begin{aligned} &= I + I' \\ &= (1 + e) I \\ &= (1 + e) \frac{m m' (u - u')}{m + m'}. \end{aligned}$$

This could also have been obtained from either of the expressions $m(u - v)$, or $m(v' - u')$, by substituting the values of v and v' just obtained.

220. *Two spheres A and A', of elasticity e and mass m and m', impinge obliquely, with velocities u and u', in directions making angles α and α' respectively with the line of impact; to determine the velocity and direction after impact.*

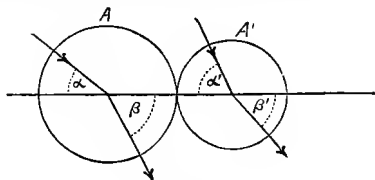


FIG. 153.

Let v and v' be the velocities after impact, β and β' the angles which their directions make with the line of impact;

and suppose that α , α' and β , β' are measured as in the accompanying figure— α and α' above the line of impact from left to right, β and β' below the line of impact from right to left.

Then we have four unknown quantities, v , v' , β , β' , to determine which we must have four equations, which are obtained as follows:—

(1) Velocity of m $\perp^r$ to the line of impact remains unchanged.

(2) " " " " " " " " " " " "

(3) Newton's law for the relative velocity in the line of impact.

(4) Total momentum in the line of impact remains unchanged.

Thus,

$$v \sin \beta = u \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$v' \sin \beta' = u' \sin \alpha' \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$v \cos \beta - v' \cos \beta' = -e(u \cos \alpha - u' \cos \alpha') \quad (3)$$

$$m v \cos \beta + m' v' \cos \beta' = m u \cos \alpha + m' u' \cos \alpha' \quad (4)$$

From equations (3) and (4), as in Article 217, we get $v \cos \beta$ and $v' \cos \beta'$; thus:

$$(m + m') v \cos \beta = (m - e m') u \cos \alpha + m' (1 + e) u' \cos \alpha' \quad (5)$$

$$(m + m') v' \cos \beta' = m (1 + e) u \cos \alpha + (m' - e m) u' \cos \alpha' \quad (6)$$

From equations (1) and (5) we obtain $v \sin \beta$ and $v \cos \beta$; squaring and adding we obtain v^2 ; dividing one by the other we obtain $\tan \beta$. Similarly we obtain v'^2 and $\tan \beta'$.

The results of the last article hold also in this case, when $u \cos \alpha$, $u' \cos \alpha'$, $v \cos \beta$, $v' \cos \beta'$ are substituted for u , u' , v , v' respectively.

Thus, the whole impulse due to the impact

$$= (1 + e) \frac{m m' (u \cos \alpha - u' \cos \alpha')}{m + m'}.$$

Example.—Spherical masses of 2 lbs. and 3 lbs., moving respectively with velocities of 2 feet per second and 3 feet per second in opposite directions, impinge directly. If the coefficient of elasticity be $\frac{1}{2}$, find the velocities after impact.

Let v and v' be the velocities after impact, measured in the direction of the 2 lbs.

Then,

$$v - v' = -\frac{1}{2}(2 + 3),$$

$$2v + 3v' = 2 \times 2 - 3 \times 3;$$

$$\therefore 2v - 2v' = -5,$$

$$2v + 3v' = -5;$$

$$\therefore v' = 0,$$

$$v = -\frac{5}{2}.$$

and

Hence the larger mass is reduced to rest, and the velocity of the 2 lbs. is $2\frac{1}{2}$ feet per second in the opposite direction to that in which it was moving before impact.

Example.—Two balls are moving with equal momenta in parallel but opposite directions, making angles of 30° with the line of impact, the mass of one being 3 lbs. and the velocity 2, and the mass of the other being 2 lbs. with velocity 3. If the elasticity be

$\frac{1}{\sqrt{3}}$, find the velocity and direction of each after impact.

Let v, v' be the velocities whose directions make angles ϕ, ϕ' with the line of impact, the measurements being taken as indicated by the figure.

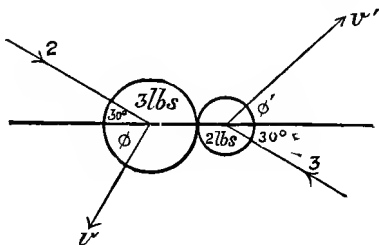


FIG. 154.

Then $v \sin \phi = 2 \sin 30^\circ = 1 \quad \dots \dots \dots (1)$

$v' \sin \phi' = 3 \sin 30^\circ = \frac{3}{2} \quad \dots \dots \dots (2)$

$$v \cos \phi + v' \cos \phi' = e (2 \cos 30^\circ + 3 \cos 30^\circ)$$

$$= \frac{1}{\sqrt{3}} \left(5 \frac{\sqrt{3}}{2} \right) = \frac{5}{2} \quad \dots \dots \dots (3)$$

$$3v \cos \phi - 2v' \cos \phi' = -3 \times 2 \cos 30^\circ + 2 \times 3 \cos 30^\circ$$

$$= 0 \quad \dots \dots \dots (4)$$

Multiply equation (3) by 2 and add it to (4), and we get

$$v \cos \phi = 1 \quad \dots \dots \dots (5)$$

$$\therefore v' \cos \phi' = \frac{3}{2} \quad \dots \dots \dots (6)$$

From (1) and (5) $v' = \sqrt{2}, \phi = 45^\circ$;

and from (2) and (6) $v' = \frac{3\sqrt{2}}{2}, \phi' = 45^\circ$.

221.* To find the motion of an elastic ball after impact with a rough plane.

In this case there is a frictional impulse perpendicular to the line of impact which diminishes the momentum of the ball perpendicular to the line of impact; and this frictional impulse $= \mu \times$ (total impulse in the line of impact), where μ = coefficient of friction.

Using the notation of Article 215, we have the frictional impulse $= \mu m u \cos \theta (1 + e)$,

$$\therefore m v \sin \phi = m u \sin \theta - \mu m u \cos \theta (1 + e);$$

$$\therefore v \sin \phi = u \{\sin \theta - \mu (1 + e) \cos \theta\};$$

and

$$v \cos \phi = e u \cos \theta;$$

$$\therefore \tan \phi = \frac{1}{e} \{\tan \theta - \mu (1 + e)\}.$$

222.* To find the motion of the centre of mass of a number of masses moving with different velocities in a plane.

Let m, m_1, m_2 , &c., be the masses.

Let Ox, Oy be two lines at right angles to each other through a fixed point O in the plane.

Let the co-ordinates of m, m_1 , &c., referred to Ox, Oy as axes, be $(x, y), (x_1, y_1)$, &c., at the beginning of time t , and $(x', y'), (x'_1, y'_1)$, &c., at the end of time t .

Let u, u_1, u_2 , &c., be their velocities parallel to Ox ; v, v_1, v_2 , &c., be their velocities parallel to Oy ; and let $\bar{x}, \bar{y}; \bar{x}', \bar{y}'$; $\bar{u}, \bar{v}$ denote the same quantities for the centre of mass.

Then, considering only the motion parallel to Ox , we have—

$$x' - x = u \cdot t,$$

$$x'_1 - x_1 = u_1 \cdot t,$$

$$x'_2 - x_2 = u_2 \cdot t,$$

and so on;

lastly,

$$\bar{x}' - \bar{x} = \bar{u} \cdot t.$$

Further, by Article 93,

$$\bar{x} = \frac{\Sigma (m x)}{\Sigma (m)}, \text{ and } \bar{x}' = \frac{\Sigma (m x')}{\Sigma (m)};$$

$$\begin{aligned} \therefore \bar{x}' - \bar{x} &= \frac{\Sigma (m x') - \Sigma (m x)}{\Sigma (m)} \\ &= \frac{\Sigma \{m (x' - x)\}}{\Sigma (m)}; \end{aligned}$$

$$\therefore \bar{u} \cdot t = \frac{\Sigma \{m u t\}}{\Sigma (m)} = \frac{t \cdot \Sigma (m u)}{\Sigma (m)},$$

since t is the same for all;

$$\therefore \bar{u} = \frac{\Sigma (m u)}{\Sigma (m)}.$$

Similarly,
$$\bar{v} = \frac{\Sigma (m v)}{\Sigma (m)}.$$

By the third law of motion $\Sigma (m u)$ and $\Sigma (m v)$ are not altered by any forces within the system—that is to say, some of the particles may be connected by strings, others may impinge against each other, others may explode, yet there is no change in $\Sigma (m u)$ or $\Sigma (m v)$, consequently $\bar{u}$ and $\bar{v}$ remain unchanged. Hence the statement, *that the motion of the centre of mass of a system of moving particles remains uniform so long as no force acts which is external to the system.*

An exactly similar proposition holds with reference to the acceleration in any given direction of the centre of mass of a number of accelerated particles; thus

$$\bar{a} = \frac{\Sigma (m a)}{\Sigma (m)}.$$

Example.—Masses m and m' are connected by a light string passing over a smooth pulley; to determine the acceleration of the centre of mass.

Let a, a' be the accelerations of m and m' respectively; then measuring positive accelerations downwards, we have

$$a = -a' = \frac{(m - m')g}{m + m'};$$

$$\begin{aligned}
 \therefore \bar{a} &= \frac{m a + m' a'}{m + m'} \\
 &= \frac{m(m - m') - m'(m - m')}{(m + m')^2} \cdot g \\
 &= \left(\frac{m - m'}{m + m'} \right)^2 g.
 \end{aligned}$$

This is positive, and therefore downwards.

Examples.

1. A person of 10 stone weight is skimming along smooth ice with a stone of $3\frac{1}{2}$ lbs. weight in his hand; he throws it in front of him, so that it leaves him with a relative velocity of 20 feet per second; by how much will his own velocity be diminished?

2. A ball drops from a height of 50 feet, and after striking the ground rebounds. Determine the height to which it will rise, the coefficient of elasticity being e .

3. A glass ball, whose coefficient of restitution is $\frac{5}{8}$, is dropped from a height of 4 feet on a stone pavement, and rebounds. How long will it be before it again strikes the ground?

4. A perfectly elastic body whose mass is 10 lbs., moving 9 yards per second, comes into direct collision with another whose mass is 11 lbs., moving 8 yards per second in the opposite direction. Find the initial velocities of each after the collision.

5. Two perfectly elastic balls of given mass are moving with given velocities in opposite directions such that their momenta are equal. Find the velocities after impact.

6. A ball of elasticity e is projected vertically upwards with a velocity V ; it falls and rebounds. Find the whole space it will have travelled when it again reaches the ground, and also when it finally comes to rest.

7. A ball moving with velocity u impinges directly upon another ball at rest whose mass is n times that of the first ball. If the first ball is reduced to rest, find the coefficient of elasticity and the rate at which the second ball will begin to move.

8. Give a geometrical construction to show the direction in which a billiard ball must be propelled in order that, after striking a perfectly elastic cushion, it may strike a ball at another part of the table.

9. A ball is projected directly against another ball lying at rest with such a velocity that, after the impact, it returns with a velocity double that imparted to the other ball. If $e = \frac{5}{6}$, find the ratio of the masses of the balls.

10. A smooth ball of mass m moving on a horizontal plane strikes another ball of mass m' at rest, and is itself reduced to rest. Find the coefficient of elasticity.

11. Two perfectly elastic balls of 4 oz. and 6 oz., moving with velocities of 6 feet and 4 feet per second, impinge directly. Find the momentum of each ball after impact.

12. A sphere of mass m impinges obliquely on a sphere of mass m' which is at rest. Show that if the modulus of elasticity be $\frac{m}{m'}$ they will separate in directions which are at right angles to each other.

13. Find the angle at which an imperfectly elastic ball must strike a plane in order that its direction of motion after impact may be at right angles to its direction before impact.

14. A ball of mass m is projected with a velocity $V = \sqrt{ga}$ from a point A up a smooth plane AB inclined to the horizon at an angle of 30° , and at the same moment a second ball of mass m' begins to move from rest from B towards A. If $AB = a$, and the elasticity of the balls $= \frac{m - m'}{2m'}$, find the velocity of each after impact.

15. An elastic particle drops from a height h on a smooth plane of inclination α ; if e be the coefficient of elasticity, find the distance between the first and second points where it strikes the plane.

16. A perfectly elastic particle is let fall from a height of 50 feet above the ground, and when half way down strikes a smooth surface inclined at an angle of 45° to the horizontal, and rebounds. At what distance from the original line of descent will the particle strike the ground?

17. A body of elasticity e is projected with velocity u from a point in a horizontal plane in a direction making an angle α with the plane; find the time that will elapse before it ceases to rebound.

18. A set of five balls of elasticity $\frac{1}{2}$, whose masses form a series

in geometrical progression whose common ratio is 2, is arranged in a straight line. The smallest impinges directly on the first with velocity u . Find the velocity of the last ball after the fourth impact.

19. A ball ($e = \frac{1}{2}$) is thrown against a smooth vertical wall from a point whose distance from the wall is u . Find the relation which must exist between the velocity of projection and the angle of projection in order that it may return to the starting point.

20. Two equal imperfectly elastic balls, moving in opposite directions along the lines BA, CA, the velocity of one being double that of the other, impinge at the same moment on a third equal ball at rest at A. Find the subsequent motion of the ball at A.

21. A ball of mass m and velocity u strikes two other balls of equal mass in contact with each other at the same instant. Show that if $e = \frac{2}{3}$ the first ball will be brought to rest, and find the initial velocity of the other two.

22. A smooth sphere slides down between a vertical wall and an inclined plane (45°) which can move along a smooth horizontal plane. If the masses of the sphere and inclined plane be equal, show that their accelerations are each equal to half that due to gravity.

CHAPTER XVII

WORK—ENERGY

223. WORK has already been defined in Chapter X.

A force is said to do work when its point of application is displaced in the direction in which the force acts ; and the work done by a force is measured by the product of the force and the projection of the displacement of its point of application in the direction of the force.

Thus, if F be the force, s the displacement measured in the direction of F , then $F \cdot s$ is the measure of the work done by the force F .

If the displacement be in the opposite direction to that of F , s is negative, and the work done is negative ; work is then said to be done *against* the force. (Article 137.)

E.g. a body of mass m falls to the ground through a height h ; then mg is the measure of the force of gravity which acts on the body, h is the distance through which the force acts ; therefore mgh is the measure of the work done by gravity on the body. If the body had been raised from the ground to the height h , the displacement would be in the opposite direction to that of the force of gravity, and mgh would be a measure of the work done against gravity.

224. To measure work we take as our unit the work done by the unit force acting through the unit distance.

The unit of force usually adopted by engineers in this country is the gravitation unit *one pound weight* ; and the unit of work is the work done by a force of 1 lb. wt. in acting through a distance of 1 foot.

This is called a *foot-pound* ; thus the work done by an agent which raises a mass of 1 lb. through 1 foot in opposition to gravity is one foot-pound.

This is the gravitation unit of work, and depends on the value of g at the place where it is adopted.

The absolute unit of work is a *foot-poundal*—that is, the work done by a *poundal* in acting through a distance of one foot ; thus, 1 foot-pound = g foot-poundals.

The unit of work adopted in France is the work done by a force of 1 dyne in acting through a distance of 1 cm. This is called an *erg*.

Since a poundal = 453.59×30.4797 dynes,

and a foot = 30.4797 cm.,

1 foot-poundal = 421390 ergs approx.

225. *To find the work done in pulling a body along a rough horizontal plane.*

If F be the limiting friction, s the distance through which the body is moved, the work done = $F \cdot s$.

Now $F = \mu R$, if R be the pressure on the plane,
 $= \mu W$, if W be the weight of the body ;

$\therefore$ the work done = $\mu W s$.

26. *To find the work done in pulling a body up a rough inclined plane of length l .*

Let h be the height, b the base of the plane.

By Article 134 the force which is required to move it up must be just greater than $W(\sin \alpha + \mu \cos \alpha)$.

Hence the work done

$$= W(\sin \alpha + \mu \cos \alpha) \times l$$

$$= W \cdot l \sin \alpha + \mu W \cdot l \cos \alpha$$

$$= W \cdot h + \mu W \cdot b$$

$$= \text{work done in lifting } W \text{ up a vertical distance } h$$

$$+ \text{work done in dragging } W \text{ along the rough horizontal surface } b.$$

227. *Definition.*—The rate at which a machine or agent can do work is called its **power**.

In estimating the capabilities of different machines of doing work we have to compare the rate at which they can do work with some standard rate of doing work. The standard adopted by engineers in England is a **horse-power**.

Definition.—A horse-power is the power of doing work at the rate of 33,000 foot-pounds per minute, or 550 foot-pounds per second.

This is rather greater than the rate at which the average horse can do work.

Example.—A man of 11 st. weight can climb a vertical height of 3,000 feet in $1\frac{1}{2}$ hour. If he is put to the treadmill, compare his rate of doing work with a horse-power.

He can raise 11×14 lbs. through 3,000 ft. in 90 minutes;

$\therefore$ he does $\frac{11 \times 14 \times 3,000}{90}$ foot-pounds of work per minute;

$$\therefore \frac{\text{the man's power}}{\text{a horse-power}} = \frac{11 \times 14 \times 3,000}{90 \times 33,000} = \frac{7}{45}.$$

228. Energy is the capacity of doing work.

When work is spent on a body there is an increase of energy in the body and a corresponding loss of energy in the agent; and when an agent does work it loses some of its energy by parting with it to the body on which it is working.

Energy may be made up of two kinds—*kinetic*, or *potential*.

Kinetic energy is the energy which a body has by virtue of its motion.

Potential energy is the energy which a body has by virtue of its position.

E.g. a body of mass m , moving with a velocity v , has kinetic energy, because in being brought to rest it is capable of doing work; and the work which it can do before it is brought to rest is a measure of the energy which it has by virtue of its motion.

The kinetic energy of a moving body is measured by $\frac{1}{2}$ (momentum $\times$ velocity), or one-half of the product of the mass into the square of the velocity—*i.e.* $\frac{1}{2} m v^2$.

The reason for this will appear in the next article. Kinetic energy is sometimes called *vis viva*.

As examples of the energy due to the position of a body we may take a stone lifted to a certain height above the surface of the earth,—the weight of a clock which has to be wound up to keep it going,—a reservoir of water which can be made to

work machinery producing mechanical power or heat energy or electric energy.

Let us consider the simplest case—a stone of mass m lifted by a man to a height h above the earth.

The work done on the stone against gravity is measured by mgh . This work is done by the man, who consequently expends the energy which the stone gains as potential energy. So long as the stone remains in this position it retains this potential energy, but when it begins to descend it can be made to do work. The quantity of work which it can do cannot be greater than mgh ; if it descends freely it does no work in its descent, but, although it loses potential energy, it gains in kinetic energy, and the total kinetic energy which it acquires just before striking the earth is a measure of the work which it can then do, *or* of the potential energy which it had before it commenced to fall, *or* of the work which was originally done upon it by the man.

229. *To find an expression for the work done on a body in terms of the mass of the body, and its velocity at the beginning and end of the action of the force.*

Work done = $F \cdot s$.

Let m be the mass of the body, u and v its velocity at the beginning and end of the action of F .

Then $F = ma$; $2as = v^2 - u^2$;

$$\begin{aligned}\therefore \text{work done} &= mas \\ &= \frac{1}{2} m (v^2 - u^2) \\ &= \frac{1}{2} m v^2 - \frac{1}{2} m u^2 \\ &= \text{change of kinetic energy.}\end{aligned}$$

Note.—In this and the following articles F is measured in absolute units.

If v is greater than u , the work done by F on m is measured by the increase of kinetic energy; if v is less than u , the work done by m against the force is measured by the loss of kinetic energy; if $u = 0$, the work done on m is measured by the kinetic energy which it acquires; if $v = 0$, the work done by

m is measured by the kinetic energy which it had before the force began to bring it to rest.

Example.—A train travelling with velocity u on level rails suddenly shuts off steam, and comes to rest after traversing a distance s . If the resistance to motion be entirely due to friction, find the coefficient of friction.

Let F = force of friction,
 m = mass of the train ;

then the work done = the kinetic energy destroyed.

$$F \cdot s = \frac{1}{2} m u^2 ;$$

but

$$F = \mu m g ;$$

$$\therefore \mu m g s = \frac{1}{2} m u^2 ;$$

$$\therefore \mu = \frac{u^2}{2 g s} ;$$

e.g. u = 60 miles an hour = 88 ft. per second,
 s = 1 mile = 5,280 ft.

$$\begin{aligned} \mu &= \frac{88 \times 88}{2 \times 32 \times 1760 \times 3} \\ &= \frac{11}{480}. \end{aligned}$$

230. When a body of mass m is raised from the ground to a height h , the work done against gravity is $m g h$.

This is the measure of the potential energy which m now possesses. That is to say, by virtue of its position it is capable of doing $m g h$ units of work at some future time.

Suppose, now, that m be allowed to fall freely from the height h to the ground ; let v be the velocity with which it reaches the ground ; then its kinetic energy

$$\begin{aligned} &= \frac{1}{2} m v^2 \\ &= \frac{1}{2} m \cdot 2 g h \\ &= m g h ; \end{aligned}$$

that is, the energy which it now has by virtue of its motion is equal to the energy which it had before by virtue of its position.

231. A body of mass m falls to the ground from a height h ; to show that the sum of its potential and kinetic energies is constant throughout the motion.

Let us consider its energy at a height x from the ground, when its velocity is v ; then its potential energy $= m g x$, and its kinetic energy

$$\begin{aligned} &= \frac{1}{2} m v^2 \\ &= \frac{1}{2} m \cdot 2 g (h - x); \end{aligned}$$

$\therefore$ the sum of its potential and kinetic energy

$$\begin{aligned} &= m g x + m g (h - x) \\ &= m g h. \end{aligned}$$

This is a simple example of the principle of the conservation of energy.

After m has been reduced to rest by contact with the ground, it has neither potential energy nor kinetic energy. What, then, has become of its energy? It has been used up in molecular change, in compressing the particles when it struck the earth, and in an addition of heat-energy to itself and the immediate neighbourhood. In this form its energy is not available for practical purposes; it is, therefore, said to have deteriorated.

Similarly, in Article 225, the work done on W is

$$W \cdot h + \mu W b;$$

this ought all to be present in the form of an increase of energy, but the only *visible* increase of energy is the potential energy $W \cdot h$; the remainder has been converted by friction into heat energy, partly in W and partly in the surface which it has traversed.

232.* When two imperfectly elastic balls impinge directly, the kinetic energy after impact is less than the kinetic energy before impact.

Let E , E' be the kinetic energy of the two balls before and after impact respectively; then, with the notation of Article 218,

$$\begin{aligned} E &= \frac{1}{2} m u^2 + \frac{1}{2} m' u'^2, \\ E' &= \frac{1}{2} m v^2 + \frac{1}{2} m' v'^2; \end{aligned}$$

$$\begin{aligned}
\therefore E' &= \frac{m}{2} \left\{ \frac{mu + m'u' - em'(u - u')}{m + m'} \right\}^2 + \frac{m'}{2} \left\{ \frac{mu + m'u' + em(u - u')}{m + m'} \right\}^2 \\
&= \frac{1}{2(m + m')^2} \{ (mu + m'u')^2 (m + m') + e^2 m m' (u - u')^2 (m + m') \} \\
&= \frac{(mu + m'u')^2 + e^2 m m' (u - u')^2}{2(m + m')} \\
&= \frac{(mu + m'u')^2 + m m' (u - u')^2 - (1 - e^2) m m' (u - u')^2}{2(m + m')} \\
&= \frac{m^2 u^2 + m m' u^2 + m'^2 u'^2 + m m' u'^2}{2(m + m')} - \frac{(1 - e^2) m m' (u - u')^2}{2(m + m')} \\
&= \frac{(m u^2 + m' u'^2)(m + m')}{2(m + m')} - \frac{(1 - e^2) m m' (u - u')^2}{2(m + m')} \\
&= E - \frac{m m'}{2(m + m')} (1 - e^2) (u - u')^2.
\end{aligned}$$

$$\text{Hence } E = E' + \frac{m m'}{2(m + m')} (1 - e^2) (u - u')^2.$$

$$\text{Therefore } \frac{m m'}{2(m + m')} (1 - e^2) (u - u')^2$$

is the kinetic energy lost by the impact; this has been transformed into some other form of energy in which it is not so apparent.

If $e = 1$ —that is, if the balls are perfectly elastic—this expression vanishes, and $E = E'$; hence there is no deterioration of energy when perfectly elastic balls impinge.

Example.—To show that in Atwood's machine the work done by gravity in any time is equal to the kinetic energy of the system.

With the notation of Article 197, let x be the distance through which the weights have moved in time t from rest, v the velocity they have acquired, then

$$x = \frac{1}{2} a \cdot t^2, \quad v = a t,$$

and

$$a = \frac{(m - m')}{m + m'} g.$$

$$\text{Work done on } m \text{ by gravity} = m g x,$$

$$,, \quad ,, \quad m' \quad ,, \quad = -m' g x.$$

Whole work done by gravity = $m g x - m' g x$

$$= (m - m') g x = (m - m') g \times \frac{1}{2} a t^2 = \frac{(m - m')^2 g^2 t^2}{2 (m + m')}.$$

Sum of the kinetic energies of m and m'

$$= \frac{1}{2} m v^2 + \frac{1}{2} m' v^2 = \frac{1}{2} (m + m') v^2$$

$$= \frac{1}{2} (m + m') a^2 t^2$$

$$= \frac{1}{2} (m + m') \left\{ \frac{(m - m') g t}{m + m'} \right\}^2 = \frac{(m - m')^2}{2 (m + m')} \cdot g^2 t^2.$$

233. In Chapter X. we have given the general statement of the principle of the conservation of energy :

'The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible.' (Maxwell's 'Matter and Motion,' chap. v.)

We have seen that this is the case in a system consisting of the earth and a stone at a distance h from it.

It is also the case when one system does work upon another system ; the quantity of energy lost by the one system is equal to the quantity of energy gained by the other ; consequently, if the two be looked upon as one system, the total quantity of energy remains unchanged.

Newton's third law of motion says :— 'Action and Reaction are equal and opposite' ; but in a scholium to this law he estimates action and reaction by 'the product of the force into the velocity of its point of application' ; thus, the action of an agent is 'the rate at which it does work'—e.g. the power of a steam engine ; thus the third law becomes 'The energy of a conservative system is constant,' or,

Work done on the system = change of energy in the system.

234. As an example of the transformation of energy we may consider the transformation of mechanical energy into heat energy.

Thus Dr. Joule, of Manchester, by a series of careful experiments, determined that in order to raise the temperature of 1 lb. of water from 39° F. to 40° F. it was necessary to expend 772 foot-pounds of work.

The above is called the unit of heat, or a thermal unit.

Hence 1 unit of heat = 772 foot-pounds of work
= 24,858 foot-pounds.

Thus, in Article 232, if the whole loss of kinetic energy is converted into heat energy, the quantity of heat gained by the impact

$$= \frac{m m' (1 - e^2) (u - u')^2}{2 (m + m') \times 24,858} \text{ thermal units.}$$

Examples.

1. Find the work done by gravity upon a stone having a mass of $\frac{1}{2}$ lb. during the tenth second of its fall from rest.

2. What is the work done against gravity by a man of 10 stone in climbing a mountain half a mile high?

3. How many foot-pounds of work can be obtained by bringing to rest a body with a mass of 1 lb. moving with a velocity of 40 feet per second?

4. A shot of $\frac{1}{2}$ ton is travelling at the rate of 1,600 feet per second; find its energy in foot-pounds, and determine for how long a time this energy, if stored, would yield two horse-power.

5. A 10 lb. mass falls 100 feet; what is the time of its fall, and what is its kinetic energy at the moment it touches the ground?

6. A lump of clay of 10 lbs. is thrown with a velocity of 50 feet per second against an equal lump at rest. If the two travel together with a velocity of 25 feet per second, find the loss in energy, estimated in foot-pounds.

7. Two bodies have the same momentum; the mass of the first is double that of the second; show that the kinetic energy of the second is double that of the first.

8. A mass of 1 lb. is dropped from a height of 120 feet; what will be its velocity when its potential energy is 56 foot-pounds?

9. A horse capable of working at one horse-power is put to grind corn by making it pull the long arm of an axle; he walks round in a circle at the rate of 4 miles an hour. What is the force (in pounds) exerted by the horse?

10. A ball whose mass is 1 lb., moving at the rate of 12 feet a second, impinges directly on an equal ball at rest. Find the velo-

cities of the balls after impact and the amount of kinetic energy lost, the coefficient of elasticity being $\frac{1}{2}$.

11. A ball of mass A, moving with velocity u , impinges directly on another ball B whose velocity is v , which in its turn impinges on a ball of mass C at rest. Show that the sum of the kinetic energies of the three balls is unchanged by the two impacts, the balls being perfectly elastic.

12. Show that if equal forces produce the same energy they must act for times which vary as the square roots of the masses acted on.

13. A man of 12 stone weight walks up a hill 3,000 feet high in $3\frac{1}{2}$ hours. What is the average power which he exerts as compared with a horse-power.

14. A blow with an inelastic hammer-head of mass $\frac{1}{2}$ lb. drives a nail to a depth of $1\frac{1}{2}$ inch into a board, the hammer-head being reduced to rest. If the velocity of the hammer-head when it first touched the nail was 15 feet per second, find the average resistance of the board to the nail's motion.

15. A shot of mass half a ton is discharged from a gun of mass 110 tons; the gun's backward motion is checked by a constant pressure equal to the weight of 10 tons; the recoil is observed to be $6\frac{3}{16}$ feet. Show that the muzzle velocity of the shot is 1,320 feet a second ($g = 32$).

APPENDIX

Duchayla's Proof of the Parallelogram of Forces.

FOR the purposes of this proof we make the following assumptions :—

- (1) That two forces, whose directions meet, have a resultant.
- (2) That this resultant may be substituted for the two components, and *vice versâ*.
- (3) The principle of the transmissibility of force. (Art. 9.)

We shall divide the proof of the proposition into four parts, and prove that it is true—

- (1) For the *direction* of the resultant of two *equal* forces.
- (2) For the *direction* of the resultant of two *commensurable*¹ forces.
- (3) For the *direction* of the resultant of two *incommensurable* forces.
- (4) For the *magnitude* of the resultant of *all* forces.
- (1) For the *direction* of the resultant of two equal forces.

Let A B, A C represent two equal forces acting through the point A.

Complete the parallelogram A B D C; join A D.

Then A B D C is a rhombus, since A B = A C, and therefore A D bisects the angle B A C; but the resultant of two equal forces must bisect the angle between the

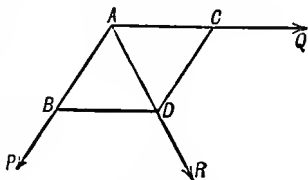


FIG. 155.

¹ Commensurable forces are forces which can be expressed as multiples of the same unit of force; if they cannot be so expressed they are incommensurable.

Thus, $2\frac{1}{3}$ and $3\frac{1}{4}$ are commensurable, but 1 and $\sqrt{2}$ are incommensurable.

Again, since it is true of mP and P , and also of mP and P , therefore it is true of mP and $2P$, and therefore of mP and $3P$, and so on; therefore it is true of mP and nP —that is, it is true for the *direction* of the resultant of commensurable forces.

(3) For the *direction* of the resultant of incommensurable forces.

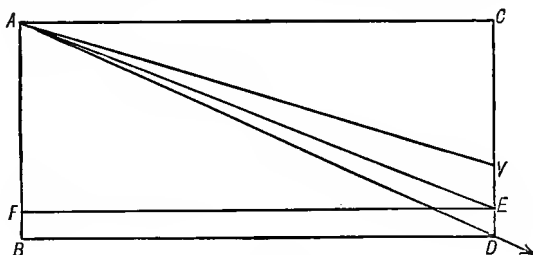


FIG. 157.

Let AB , AC represent two incommensurable forces P and Q acting at the point A . Complete the parallelogram $ABDC$; then AD shall represent the direction of the resultant of P and Q ; for if this resultant does not act in the direction AD , let us suppose that it acts in any other direction as AV , and show the absurdity of such a supposition.

Let AV cut CD in V .

Divide AC into a number of *equal* parts, each less than VD ; from CD cut off parts equal to these commencing from the end C ; the last point of division must fall between V and D , since each part is less than VD , and it cannot fall on D since AC and CD are incommensurable lines; let it fall at E . Complete the parallelogram $ACEF$, and join AE .

Now the force P is represented by AB —that is, by AF , FB , of which AF is commensurable with AC ; therefore the resultant of P and Q is the same as the resultant of forces represented by AC , AF , and FB ; now the resultant of the commensurable forces represented by AC , AF acts along AE by (2), and the force represented by FB may be supposed to act at A in the direction AB ; therefore the resultant of P and Q is in the same direction as the resultant of these two forces acting along AE and AB ; hence it *must* act within the angle EAB ; therefore the supposition that it acts along AV is absurd.

Similarly it can be shown that it is absurd to suppose that the

resultant of P and Q is in any other direction than AD ; therefore AD must be the direction of the resultant.

We have thus proved that the proposition is true for the *direction* of the resultant of all forces; (4) we will now prove that it is also true of the *magnitude* of the resultant.

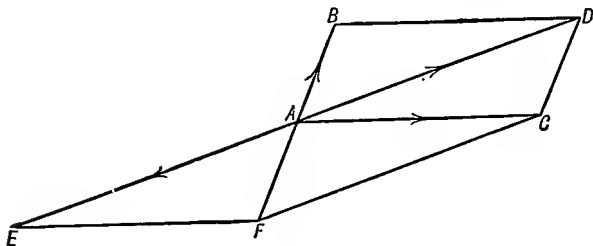


FIG. 158.

Let AB , AC represent any two forces P and Q ; complete the parallelogram $ABDC$, join AD ; then AD represents the direction of the resultant of P and Q ; it shall also represent the resultant in magnitude.

Produce DA through A to E , making AE of such a length as to represent the magnitude of the resultant of P and Q ; complete the parallelogram $EACF$, and join AF .

Then, since AE has been drawn equal in *magnitude* and *opposite* in *direction* to the resultant of P and Q , it follows that the forces represented by AB , AC , and AE must be in equilibrium, and either of them must be equal and opposite to the resultant of the other two; but AF represents the *direction* of the resultant of forces represented by AC , AE by (3), therefore AF must be in the same straight line as AB ; but AB is parallel to CD ; therefore FA is parallel to CD , and FC is parallel to EAD ; therefore $FADC$ is a parallelogram; therefore $AD = FC$; but $FC = AE$; therefore $AD = AE$; but AE represents the magnitude of the resultant of P and Q ; therefore AD also represents the magnitude of the resultant of P and Q .

Hence, if any two forces acting at a point be represented in magnitude and direction by two lines drawn from that point, and if the parallelogram having these two lines for adjacent sides be completed, the diagonal of this parallelogram, which passes through the point, represents the resultant of the two forces both in *magnitude* and *direction*.

ANSWERS

CHAPTER I.

1. 88. 2. $5\frac{1}{2}$. 3. 28·16. 4. $111\frac{1}{5}$. 5. $\frac{60a}{b}, \frac{300b}{a}$.
 6. 1,536 ft. per sec. 7. 10 miles an hour. 8. 5 ft. per sec. per sec.
 9. 4 ft. per sec. 10. $\frac{1}{2}$ the unit of velocity is added every second.
 11. 9·8. 12. 10; 10 poundals; 1,000 poundals; $\frac{1000}{g}$. 13. 200,000:1.
 14. 2,000. 15. 5; 300. 16. 80. 17. 560. 18. Acceleration = 5;
 mass = 5 tons.

CHAPTER III.

1. 4, 2, or 0. 2. 14 lb. weight. 3. 30 lb. weight. 4. 12, 2;
 3 and 4 at right angles. 5. 101. 6. $\sqrt{5-2\sqrt{2}}$. 7. 7.
 8. $\sqrt{37}$. 9. 15. 10. $120^\circ, 150^\circ, 90^\circ$. 11. $10\sqrt{3}, 20$.
 12. $P\sqrt{7}; \sin^{-1}\left(\frac{\sqrt{21}}{7}\right)$ with P. 13. 13. 18. $\sqrt{3}:\sqrt{2}$.
 19. $P\sqrt{5}; \tan^{-1}\frac{1}{2}$ with resultant. 20. $\frac{10\sqrt{3}}{3}, \frac{20\sqrt{3}}{3}$.
 21. $\cos^{-1}\left(-\frac{1}{5}\right)$. 22. $\cos^{-1}\left(-\frac{11}{16}\right)$. 23. $\cos^{-1}\frac{1}{5}$. 24. $\cos^{-1}\left(\frac{1}{26}\right)$.
 25. $P = 2\sqrt{2}, Q = 2$. 26. $\sqrt{2}:1$. 27. $S = P$ and makes an
 angle of 60° with it. 28. 3 R. 29. P. 34. $\frac{P\sqrt{5}}{5}$.
 35. The diameter through A. 39. $\sin^{-1}\frac{1}{\sqrt{5}}$ with the side.
 41. $\sqrt{(P-R)^2 + (Q-S)^2}$. 42. AB or DC. 43. AC.
 44. 4, $2\sqrt{3}$. 46. 6 in the direction of the force 5. 47. 13.
 48. 1 opposite to the direction of 19. 49. 10 opposite to the direction
 of 8. 50. $\cos^{-1}\left(\frac{13}{15}\right)$. 51. 150° . 52. 2 AC. 53. 1:2.

54. $\frac{7}{2}, \frac{7\sqrt{3}}{2}$. 55. $\frac{3}{2}, \frac{3\sqrt{3}}{2}$. 56. 6 lb. weight. 57. 5 A B
 along A D. 58. $\sqrt{29 - 18\sqrt{2}}$. 59. $\sqrt{105 + 42\sqrt{3}}$ at an angle
 $\tan^{-1}\left(\frac{7 - \sqrt{3}}{2}\right)$ with the force 8. 60. $\sqrt{41}$. 64. $\frac{24}{25}$. 65. When
 the given angle θ is obtuse and the given component is greater than the
 given force but less than $P \operatorname{cosec} \theta$. 68. $\frac{8\sqrt{29}}{3}$.

CHAPTER IV.

1. 6 ins. from 7. 2. 12 ins. from 9. 3. 4 ft. 4. 1 ft.
 5. 5, $7\frac{1}{2}$. 6. $1\frac{1}{5}$ ft. 7. $3\frac{1}{2}, 1\frac{1}{2}$. 9. $1\frac{5}{8}$ ft. from 1 lb. 10. $3\frac{6}{7}$ ft.
 11. 160 lb. wt. 12. 2 lb. wt. 13. $5\frac{1}{4}$. 14. $\frac{1}{5}$. 15. $85\frac{1}{2}, 85\frac{1}{2}, 58$.
 16. At the 3 lb. wt. 18. $7\frac{1}{2}$ lb. wt. 19. $\frac{2}{5}$ in. from A.
 21. $\frac{2(n+2)}{3}$ ins. 22. $A O = \frac{5}{6} A B$. 24. $P : Q : R :: a^2 : b^2 : c^2$.

CHAPTER V.

2. 2 : 1. 5. The same. 6. $1\frac{1}{8}$ ft. from 7. 7. Each is 24.
 8. 12 P, 9 P, $\frac{36}{5}$ P. 9. 1 : 4. 13. 90° . 14. 1 ft. from one end.
 15. 2 oz. 16. $1\frac{1}{2}$ in. 18. A couple whose moment is 6 a .

CHAPTER VI.

1. $1\frac{1}{3}$ in. from the 5 lbs. 2. $\frac{5}{9}$ in. from 8 W. 3. $\frac{2}{5}$ (diagonal)
 from the 2 lbs. 4. $\frac{4}{7}$ (diagonal) from the 1 lb. 6. $1\frac{11}{12}$ ft. from 3 lbs.
 10. 10 ins. 12. It coincides with that of the triangle A B C.
 13. $\frac{23}{48}$ B D from D. 14. If O be the centre of the board, G the centre
 of gravity required, $O G = \frac{\sqrt{2}}{14}$ radius. 16. $\frac{1}{3}$ (radius) from the centre.
 17. 1 lb. wt. 19. It will balance at O if $A O = \frac{1}{2} A G$. 20. 60° .
 21. 3. 23. It divides the diagonal in the ratio 11 : 13.
 24. $O G = \frac{2}{21} O C$. 25. $\frac{3a}{8 + 2\sqrt{3}}$ from the base. 26. It divides
 the diagonal of the larger square in the ratio 7 : 13. 27. It divides the
 perpendicular from the centre in the ratio 2 : 7. 28. $11\frac{1}{2}$ ins.
 29. $\sqrt{3} : 1$. 32. $O G = \frac{1}{4} O D$. 33. It divides the perpendicular
 from the right angle upon the hypotenuse in the ratio 26 : 1. 34. It
 bisects the line joining A to the middle point of B C. 35. 16 ins. from

- the centre. 37. $\tan^{-1} \frac{8}{3}$ and $\tan^{-1} \frac{3}{4}$. 38. $\frac{1}{2} a$ from A D, $\frac{2}{3} a$ from A B. 39. 6 lb. wt. 40. 12 lb. wt. 42. The sides 3 and 4 are at right angles to each other; $2\frac{1}{2}$, $2\frac{5}{8}$. 44. It will be on the point of falling. 45. $3\frac{11}{13}$ ft. 46. $\frac{\sqrt{3}-1}{2}$ (side of square). 47. 8 : 15. 48. $\frac{63}{88} < \frac{3}{4} < \frac{19}{24}$. 49. Its distances from A D and A B are $\frac{11}{18}$ A B and $\frac{4}{9}$ A D. 50. $\frac{7a^2b - 4b^3 + 3\sqrt{3}a^2\sqrt{4a^2 - b^2}}{3(2\sqrt{3}a^2 + \sqrt{3}b^2 + b\sqrt{4a^2 - b^2})}$; if $b = a$ this = $\frac{a}{\sqrt{3}}$.

CHAPTER VII.

1. 1, $\sqrt{3}$. 3. $\sin \theta = \frac{a}{\sqrt{a^2 + b^2}}$. 5. The longer arm is inclined at an angle $\tan^{-1} \frac{7\sqrt{3}}{3}$ to the horizon. 6. $\sin^{-1} \sqrt{\frac{a}{l}}$. 7. $\frac{W\sqrt{3}}{3}$. 8. $\frac{W}{\sqrt{3}}$. 9. $18\frac{2}{3}$ ft., $1\frac{2}{3}$ lb. wt. 10. $W(\sqrt{2} - 1)$. 12. 2 lb. wt. 13. 91 lb. wt. 14. $\frac{\sqrt{3}}{3}$ ton. 15. $\frac{22}{3}$ st. wt.; 0. 16. $\sqrt{146}$. 17. $2a$, $2W$. 18. First force = W and makes an angle $90^\circ - 2\theta$ below the horizon; the other force = $2W \cos \theta$. 19. $\tan \theta = \frac{l}{a} \cos^2 \theta$.

CHAPTER VIII.

LEVER.

1. $\frac{7}{10}$ ft. from the 3 lbs. 2. $18\frac{1}{5}$ ins. 3. $6\frac{3}{4}$ ins. 4. $\frac{4}{9}$ ft. from the 8 lbs. 5. $1\frac{5}{7}$ ft. 6. 4 lb. wt. 7. 30° . 8. 12 ins. from the 20 lbs. 9. W . 10. $9\frac{4}{5}$ lb. wt. 11. 15. 12. $\frac{9}{11}$. 13. 1 : $2\frac{1}{2}$. 14. $\frac{P^2}{Q}$. 16. $13\frac{1}{3}$ ft., 15 lb. wt. 17. 3 Q.

THE COMMON STEELYARD.

18. $5\frac{1}{2}$ ft.; 23 lbs. 20. 18 ins., $\frac{4}{3}$ lb. 21. $\frac{4}{3}$ in. from C; $2\frac{1}{2}$ lbs.; $41\frac{1}{3}$ ins. from A. 22. 16 lb. wt. 23. 5 lbs. excess. 25. $3\frac{1}{4}$ lb. wt.

THE DANISH STEELYARD.

26. 3 oz., $\frac{1}{6}$ oz. 27. 12 ins. from the end; 4 oz. wt. 28. 35 lbs. 30. $2\frac{1}{2}$ ft.

BALANCE.

31. $26\frac{1}{4}$ lbs. 32. $\frac{1}{4}d$. 35. $8\frac{32}{121}$ lbs. 36. 1 oz. 37. $3\frac{1}{2}$ lb. wt.; 7 : 8.

WHEEL AND AXLE.

38. $16\frac{2}{3}$ lb. wt. 39. 24 cwt. 40. 55 lb. wt.

PULLEYS.

41. $84\sqrt{2}$ oz. 42. $\frac{W}{\sqrt{3}}$ **First System.** 43. 4. 44. 4 lb. wt.
 45. $\frac{15}{16}w$. 46. 17 oz. wt. 47. 81 lb. wt. 48. 3 lb. wt. 49. Wt. of
 15 lb. 1 oz. 50. The middle pulley being 5 lbs.—864 or 858 lb. wt.
 51. $10\frac{15}{16}$ oz. wt. 52. 5 lb. wt. 53. 16 lb. wt. 54. 7; 32 ft. **Second**
System. 55. 440 lb. wt. 56. 12 lb. wt. 57. 15 lb. wt. 58. 5 pulleys.
 60. 4; 2 lb. wt. **Third System.** 61. 4. 62. 26 or 28 lb. wt.
 63. 14 lb. wt. 64. 1 lb. wt. 65. 16 lbs. 6 oz. 66. $\frac{15}{2}w$.
 67. 11 oz. wt. 68. 10 lb. wt. 70. 5 lb. wt.

INCLINED PLANE.

72. 6 lb. wt. 73. $12\sqrt{3}$ lb. wt. 74. 30° . 75. 45° .
 76. 20 lb. wt. 77. 30° , $6\sqrt{3}$ lb. wt. 78. 30° , 30° with the plane.
 79. 30° . 80. At an angle $\cos^{-1}\frac{2}{3}$ with the plane. 82. 45° .
 83. P makes an angle with R equal to the angle of the plane.
 84. $\sqrt{3} : \sqrt{2}$. 85. 103.92 lb. wt. 86. 5 : 3 : 4; 2 P.
 92. $\cos^{-1}\frac{3}{4}$ with the plane. 93. $W \cos \alpha \cdot \sin \alpha$. 94. 7 st. wt.

CHAPTER IX.

1. 8 lb. wt., 4 lb. wt. 2. $\frac{1}{4}$; $\sin^{-1}\frac{3}{5}$. 3. $\frac{W}{2}$. 4. $\mu = \frac{1}{\sqrt{3}}$.
 5. $P = (1) W$, (2) $\sqrt{3}W$; $R = (1) W$, (2) $2W$. 6. $\frac{4}{3}$. 8. P and R
 are at right angles. 10. Twice the angle of friction. 11. $\frac{\sqrt{3}}{5}$.

CHAPTER XI.

1. 7. 2. $\frac{1}{4}$ hour. 3. $\frac{11}{75}$. 4. 38,400. 5. 192 g.
 8. 24, 40 nearly. 7. About $7\frac{3}{11}$ miles per hour; $22\frac{1}{2}$ miles.
 8. 50 miles an hour. 9. $\frac{88}{5}$. 10. 242 ft., $2\frac{3}{4}$ sec. 11. $\frac{240}{11}$.
 12. $\frac{55}{54}$. 13. $264\sqrt{10}$. 14. 1. 15. 8, 32. 16. $8\frac{4}{5}$. 17. 1.
 18. $\frac{5g}{144}$. 20. $\frac{22}{45}$. 21. $\frac{3}{4}$ mile. 22. 10 sec. 23. $\frac{11}{225}$ in ft.-
 sec. units; 120 in mile-hour units. 24. 1 in 654. 25. $\frac{11}{15}$; $\frac{11}{15g}$.

26. $\frac{2n-1}{2n+1}$. 27. 1,200. 28. 128.8. 29. 48, 72. 30. 4,440 miles per hour per hour. 31. 1,936 ft. 32. 44 secs. 33. Acceleration = 32. 34. 34 ft. 35. Velocity = $1\frac{4}{85}$ miles per min.; acceleration = $\frac{8}{85}$ miles per min. per min. 37. 12 miles.

CHAPTER XII.

1. 253.7. 2. $\frac{5\sqrt{2}}{4}$, $40\sqrt{2}$. 3. 32.18 . 4. 20. 5. 72.
6. 576 ft. 7. 320 ft. 8. $\tan^{-1} \frac{4\sqrt{2}}{11}$. 9. 720 ft., 21 sec.
10. 4.65 sec. 11. 64 ft. 12. $6\frac{1}{4}$ sec., $312\frac{1}{2}$ ft. 13. $\sqrt{12}g$.
14. 5 sec. 15. $24 + 8\sqrt{5}$ ft. 16. 9 sec. 17. 64 ft. 19. 159.04.
20. 676 ft. 21. 175 ft. $\frac{85}{104}$ sec. 22. 30° . 23. $\frac{1}{2}$ sec. 25. $a \sec \frac{a}{2}$.

CHAPTER XIII.

2. 19,600; 58,800. 3. $2\frac{37}{84}$ miles. 4. 45° . 5. $\tan^{-1} 4$.
6. 1162.6. 7. 48; $36\sqrt{3}$. 8. $6\frac{1}{4}$ ft.; 1,500 ft. 10. $80\sqrt{3}$.
14. 200 ft. 15. $10\sqrt{3}$ sec. 16. 72; $2\frac{2}{5}$ sec. 19. 24.
22. $(2\sqrt{2} - 1)(V^2 \sin \alpha \cos \alpha)/g$ from the point of projection.
23. $\frac{2}{g}\sqrt{v^2 - gR}$. 26. $\frac{L}{\sqrt{2}}$. 31. At the lowest point. 32. $\tan \theta$
= $(\tan \alpha)^{\frac{1}{2}}$ or $(\cot \alpha)^{\frac{1}{2}} - (\tan \alpha)^{\frac{1}{2}}$.

CHAPTER XIV.

1. $\frac{1}{50}$ poundal. 2. They are equal. 3. 220 poundals.
4. $mv\sqrt{2}$ along the diagonal. 5. $\frac{g}{17}$. 6. About 2.84 secs.
7. $\frac{g}{13}$; $\frac{36g}{13}$ poundals. 8. (1) $\frac{2}{3}$; (2) $5\frac{1}{3}$; (3) $2\frac{1}{3}$. 9. $3\frac{2}{3}$; $2\frac{2}{3}$; $\frac{32}{33}$.
10. $\frac{8g}{9}$ poundals; 16 ft. 11. $4\frac{1}{2}$ oz., $3\frac{1}{2}$ oz. 12. 2.
13. $\frac{u^2}{2g}$; $\frac{2M+m}{m}$. 15. $\frac{M-m}{M+m}$ sec.; $\frac{(3M+m)(M-m)g}{2(M+m)^2}$ ft.
16. $5\frac{1}{3}$ ft.; 2 secs. 17. 17:15. 18. $\frac{98m}{225}$ poundals, where m = mass
of train in lbs. 19. $34\frac{2}{9}$ secs. 20. $\sqrt{2} - 1$. 21. $\frac{g}{10}$.

22. $2\frac{5}{8}$ miles; 7 mins. 23. $\frac{g}{4}$; $\frac{3mg}{4}$ poundals. 24. $\frac{487500}{981}$ and $\frac{10000}{981}$.
 25. 2.4. 26. $14\frac{4}{5}$. 27. $2\frac{5}{8}$ secs. 28. 4 ft. 11 ins. 29. $\frac{5}{4}$ ft. per sec.
 31. $W \cdot \frac{g t^2}{g t^2 + 2 h}$; $\frac{5 g t^2 + 4 h}{2 (g t^2 + h)} \cdot h$. 32. Acceleration of R
 $= \frac{4 P Q - R (P + Q)}{4 P Q + R (P + Q)} \cdot g$.

CHAPTER XV.

$$6. \frac{v^2}{\rho} = g \cos \theta. \quad 10. 31.92.$$

CHAPTER XVI.

1. $\frac{1}{2}$. 2. $50 e^2$. 3. $\frac{5}{8}$ sec. 4. $8\frac{17}{21}$ and $8\frac{4}{21}$ yards per sec. in the opposite directions. 5. Same as before, but in opposite directions.
 6. $\frac{V^2}{g} (1 + e^2)$; $\frac{V^2}{g} \left(\frac{1}{1 - e^2} \right)$. 7. $\frac{1}{n}$, $\frac{u}{n}$. 9. 5 : 28. 10. $\frac{m}{m'}$.
 11. $\frac{9}{16}$, $\frac{21}{16}$. 13. $\tan^{-1} \sqrt{e}$. 14. They meet with equal velocities $\frac{V}{2}$; m is at rest after impact, and m' starts up the plane with velocity $e V$.
 15. $4 e (1 + e) h \sin \alpha$. 16. 50 ft. 17. $\frac{2 u \sin \alpha}{g} \cdot \frac{1}{1 - e}$. 18. $\frac{u}{16}$.
 19. $v^2 \sin 2 \alpha = 3 a g$. 20. $\frac{u (1 + e)}{3}$. 21. $\frac{u \sqrt{3}}{3}$.

CHAPTER XVII.

1. 152 foot-pounds. 3. 25. 4. 11.3 hrs 5. $2\frac{1}{2}$ secs.; 100 mg .
 6. 6,250. 8. 64. 9. 3,000 poundals. 10. 3, 11; 27. 13. $\frac{4}{55}$.
 14. 450 poundals.

